

Math 4200

Fri 9/16

2.1 2ac, 3, 5, 10, 11, 13, 14

2.2 1ad, 2 (prove with FTC!),
3, 4, 6, 8, 11.

+ Class exercise, page 4 today's notes
(1.6 # 6, 10, 14 are carried over from last week)

Chapter 2: Complex integration

- leads to Cauchy Integral formula and magic theorems:

- Liouville: bounded entire functions are constant
- F.T. Algebra: every degree n polynomial has n (complex) roots (counting multiplicity)
- magic ways to calculate integrals (contour integration)

Definition A: For $f: [a, b] \rightarrow \mathbb{C}$ continuous, $f(t) = u(t) + iv(t)$ $u = \text{Re } f$
 $v = \text{Im } f$

then $(A1) \int_a^b f(t) dt := \int_a^b u(t) dt + i \int_a^b v(t) dt$

is equivalent to

$(A2) \int_a^b f(t) dt := \lim_{\|P\| \rightarrow 0} \sum_j f(t_j^*) \Delta t_j$

(norm of partition approaches zero)

$P: a = t_0 < t_1 < \dots < t_n = b$

$t_{j-1} \leq t_j^* \leq t_j; \Delta t_j = t_j - t_{j-1}$

$\|P\| = \max_j \Delta t_j$

$= \lim_{\|P\| \rightarrow 0} \sum_j (u(t_j^*) + iv(t_j^*)) \Delta t_j$

$= \lim_{\|P\| \rightarrow 0} \sum u(t_j^*) \Delta t_j + i \sum v(t_j^*) \Delta t_j$

$= \int_a^b u(t) dt + i \int_a^b v(t) dt$ ✓

Exercise 1 $\int_0^{\pi/2} -2 \sin t \cos t + i(\cos^2 t - \sin^2 t) dt =$

43) Fundamental Theorem of Calculus (apply real FTC to real & imag parts)

$\int_a^b f(t) dt = F(b) - F(a)$ if $F'(t) = f(t) \forall t \in [a, b]$.

$f(t) = u(t) + iv(t)$
 $F(t) = U(t) + iV(t)$
 $U' = u \ \& \ V' = v$

44) Integral estimate

$|\int_a^b f(t) dt| \leq \int_a^b |f(t)| dt$

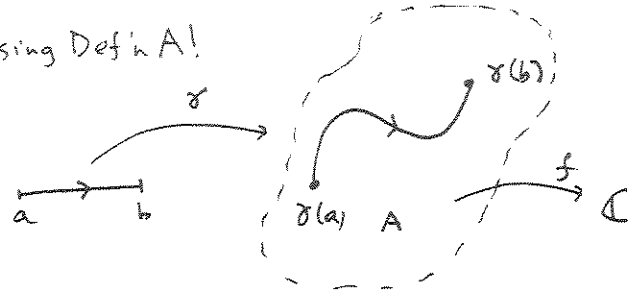
pf: apply A2: $|\int_a^b f(t) dt| = |\lim_{\|P\| \rightarrow 0} \sum_j f(t_j^*) \Delta t_j|$
 $= \lim_{\|P\| \rightarrow 0} |\sum_j f(t_j^*) \Delta t_j|$

$\leq \lim_{\|P\| \rightarrow 0} \sum |f(t_j^*)| \Delta t_j$ \triangle ineq!
 $= \int_a^b |f(t)| dt$ ■

Definition B Let $A \subset \mathbb{C}$ open, $f: A \rightarrow \mathbb{C}$ continuous (not necessarily analytic, although later on this will be our primary focus) (2)
 $\gamma: [a, b] \rightarrow A$ a C^1 curve
 $\underbrace{\quad}_{\mathbb{R}}$

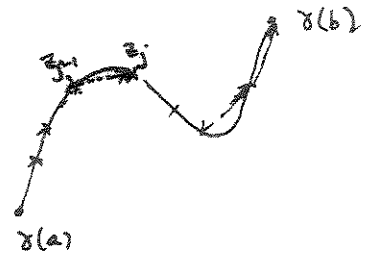
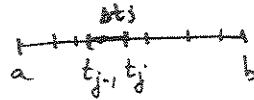
$$(B1) \int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt \leftarrow \text{using Def'n A!}$$

(so, you substitute $z = \gamma(t)$ and the differential $dz = \gamma'(t) dt$ to evaluate the contour integral)



in case $|\gamma'(t)| > 0$ on $[a, b]$ (so $|\gamma'(t)| \geq \delta > 0$), then (B1) is equivalent to

$$(B2) \int_{\gamma} f(z) dz = \lim_{\max |\Delta z_j| \rightarrow 0} \sum f(z_{j-1}) \Delta z_j$$



$$= \lim_{\max |\Delta t_j| \rightarrow 0} \sum f(\gamma(t_{j-1})) \gamma'(t_{j-1}) \Delta t_j + f(\gamma(t_{j-1})) \varepsilon(t_j) \Delta t_j$$

$$\Delta z_j = z_j - z_{j-1} = \gamma(t_j) - \gamma(t_{j-1}) = \gamma'(t_{j-1}) \Delta t_j + \varepsilon(t_j) \Delta t_j$$

$$= \int_a^b f(\gamma(t)) \gamma'(t) dt + 0 \quad \checkmark$$

where $\varepsilon(t) \rightarrow 0$ uniformly as $\Delta t \rightarrow 0$, because $\gamma \in C^1$

Exercise 2 Let $\gamma(t) = e^{it}$ $0 \leq t \leq \pi/2$

$$f(z) = z$$

$$\text{compute } \int_{\gamma} f(z) dz$$

Exercise 3 $\gamma(t) = e^{it}$ $0 \leq t \leq 2\pi$

$$f(z) = \frac{1}{z}$$

$$\text{compute } \int_{\gamma} \frac{1}{z} dz$$

(B3) FTC: If $\exists F: A \rightarrow \mathbb{C}$ analytic, with $F' = f$ on A ,

then
$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a))$$

$$\begin{aligned} \text{pf: } \int_{\gamma} f(z) dz &= \int_a^b f(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b \underbrace{F'(\gamma(t)) \gamma'(t)}_{\frac{d}{dt} F(\gamma(t))} dt = F(\gamma(t)) \Big|_a^b \quad \text{by (A3)} \end{aligned}$$

(B4) Integral estimate:

$$\begin{aligned} \left| \int_{\gamma} f(z) dz \right| &= \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \\ &\leq \int_a^b |f(\gamma(t)) \gamma'(t)| dt \quad (\text{A4}) \\ &= \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \end{aligned}$$

(B5):
$$\int_{\gamma} |f(z)| |dz| = \int_a^b |f(\gamma(t))| |\gamma'(t)| dt$$

so (B4) reads
$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz|$$

Exercise 4 Rework Exercise 2 using FTC

Exercise 5 What is $\int_{\gamma} |f(z)| |dz|$ for Exercise 2 Example?

Exercise 6: Let $\gamma(t) = e^{it}$ $0 \leq t \leq 2\pi$
 $f(z) = z^n$ $n \in \mathbb{Z}$.

What is $\int_{\gamma} f(z) dz$?

Relation of contour integrals to (real) line integrals:

write $f(z) = u(x,y) + iv(x,y)$
 $\gamma(t) = x(t) + iy(t)$

then

$$\int_{\gamma} f(z) dz = \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))] [x'(t) + iy'(t)] dt$$

$$= \int_a^b u(x(t), y(t)) x'(t) - v(x(t), y(t)) y'(t) dt$$

$$+ i \int_a^b v(x(t), y(t)) x'(t) + u(x(t), y(t)) y'(t) dt$$

$$= \int_{\gamma} u dx - v dy + i \int_{\gamma} v dx + u dy$$

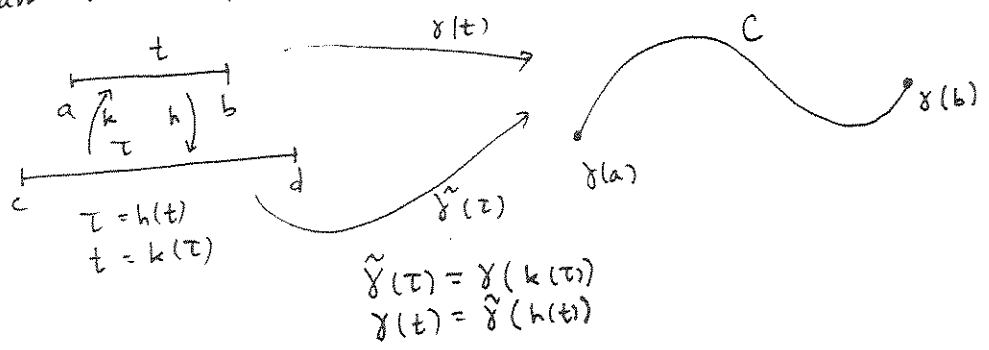
easy to derive formally by writing

$f = u + iv, dz = dx + idy$

$$\int_{\gamma} (u + iv)(dx + idy) = \int_{\gamma} u dx - v dy + i \int_{\gamma} v dx + u dy. \quad \checkmark$$

Class exercise Change of variables in complex curve integrals: Let $\gamma: [a, b] \rightarrow C \subset \mathbb{C}$
 $\tilde{\gamma}: [c, d] \rightarrow C$

have the same image curve so that the following diagram commutes, and all functions are C^1 with non-zero derivatives



Use chain rule to prove

$$\int_{\tilde{\gamma}} f(z) dz = \begin{cases} + \int_{\gamma} f(z) dz & \text{if } k(c) = a \text{ and } k(d) = b \\ - \int_{\gamma} f(z) dz & \text{if } k(c) = b \text{ and } k(d) = a. \end{cases}$$