Math 4200
Wed 14 Sept

§ 1.6 : The zoo of common analytic functions, their derivatives;
and branches of their inverses, and of compositions involving their inverses.

(we may still wish to go over the connectivity discussion on page 3 Monday.)

Def : If $f : \mathbb{C} \to \mathbb{C}$ is analytic $\forall z \in \mathbb{C}$, $f$ is called entire

examples:

$f(z) = z^n \quad (n = 0, 1, 2, \ldots)$

$f'(z) = n z^{n-1}$ (by binomial thm or product rule induction)

$f(z) = e^z$

$f'(z) = e^z$ (by direct computation, $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$
or by Cauchy Riemann + $C'$ partials)

$f(z) = \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$f'(z) =$ (chain rule)

$f(z) = \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$f'(z) =$

etc.

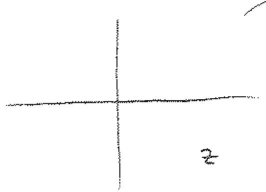
"Branches" of analytic functions : overview

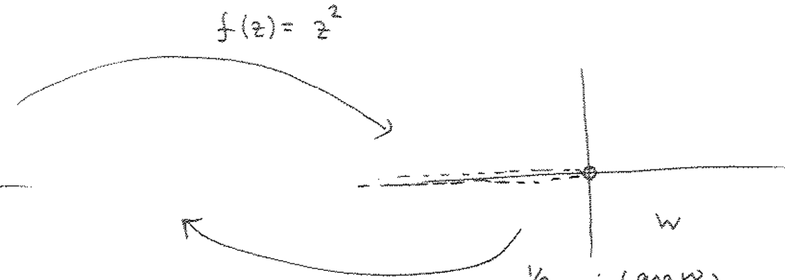
If $f$ is entire then it turns out (Picard's theorem) that if $f$ is not a constant function then, in fact, $f$ is almost onto (the precise theorem is that if the range of $f$ omits 2 or more points, then $f$ is constant → called Picard's thm.) furthermore, the zeroes of $f'(z)$ are isolated, so $f$ has local inverse fcns at all but a countable set of $z$'s.

In this case one can construct an "inverse" function $g$, which satisfies half of the inverse function condition, $f(g(z)) = z$, on a connected open domain $A$ which is $\mathbb{C}$ with a finite number of curves removed. These curves are called branch cuts, and the choice of $g$ is called a branch of the inverse "function". Branch cuts always terminate at $\infty$ or at a finite value branch point, $f(z)$, where $f'(z) = 0$. or at a point not in the range of $f$

examples (revisited)

$f(z) = z^2$




$g(w) = |w|^{1/2} e^{i\left(\frac{\arg w}{2}\right)}$ $\quad -\pi < \arg w < \pi$

(= one of the $\sqrt{w}$'s)

branch cut along the negative real ray.

(but you could use any reasonable curve from 0 to $\infty$ as the branch cut, and define a different branch of $\sqrt{w}$)

$\text{range}(g) = \{z \mid \text{Re}\, z > 0\}$
$= 1^{st} \cup 4^{th}$ quadrants

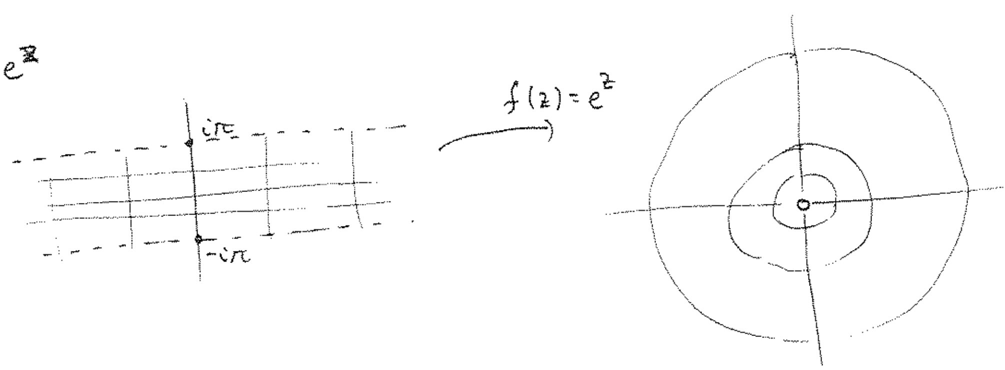
since $f(g(w)) = w$

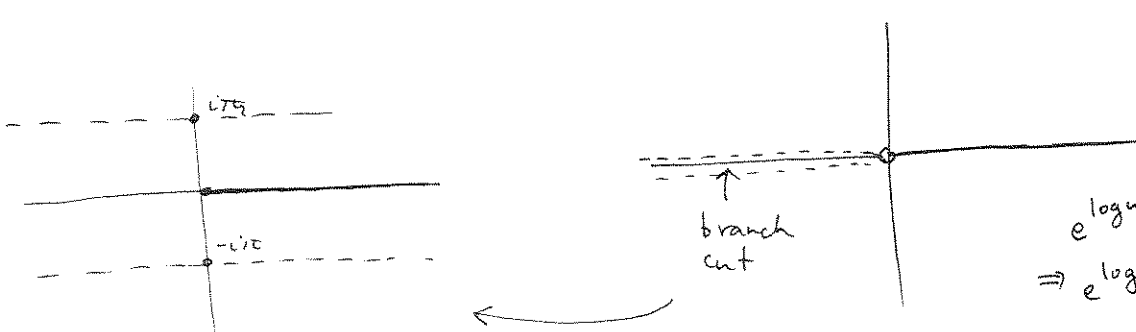
$f'(g(w))\, g'(w) = 1$

$g'(w) = \frac{1}{2\, g(w)}$

$(\sqrt{w})' = \frac{1}{2} w^{-1/2}$ , would work for any branch choice, as long as $\sqrt{w}$ is defined consistently

$f(z) = e^z$




$e^{\log w} = w$

$\Rightarrow e^{\log w} (\log w)' = 1$

$(\log w)' = \frac{1}{w}$

for any branch choice.

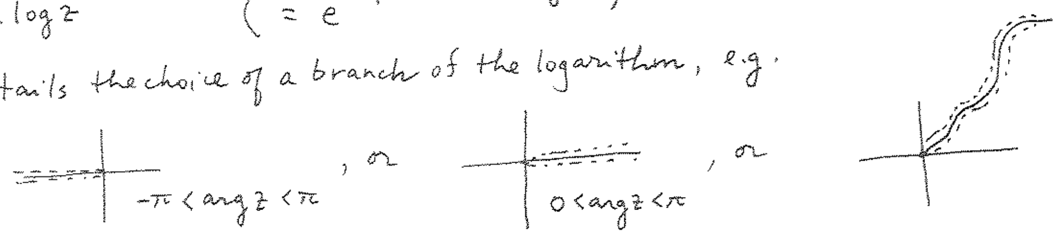
$g(w) = \log w$
$= \ln|w| + i \arg w$
$-\pi < \arg w < \pi$
"standard branch"
note, on positive x-axis restricts to $\ln x$, inverse of $e^x$

we kind of had this discussion before, except for computing the derivatives...

for any $a \in \mathbb{C}$

$$z^a := e^{a \log z} \qquad \left( = e^{a(\ln|z| + i \arg z)} \right)$$

entails the choice of a branch of the logarithm, e.g.




$$\frac{d}{dz} z^a = \frac{d}{dz} e^{a \log z} = e^{a \log z} \frac{a}{z} \quad \text{chain rule}$$

$$= z^a \frac{a}{z} = a z^{a-1}$$

(provided we use the same branch of logarithm).

(1) if we use standard branch of log, $\arg x = 0$ $(x \in \mathbb{R}^+)$ $\Rightarrow x^a = e^{a \ln x}$ agrees with Calc def.

(2) if $n \in \mathbb{Z}$, $z^n := e^{n \log z} = e^{n(\ln|z| + i \arg z)} = |z|^n e^{in\theta} = z^n$ algebraic def.

(3) if $n \in \mathbb{N}$, $z^{\frac{1}{n}} := e^{\frac{1}{n}\log z} = e^{\frac{1}{n}(\ln|z| + i\arg z)}$

$= |z|^{1/n} e^{i\frac{\theta}{n}}$ is one of the n'th roots of $z$, depending on branch of log.

More complicated fcns involving composition, where branch choices are made at one or more stages.

<u>Goal</u>: Find a domain for the given function which is a connected open subset of $\mathbb{C}$ and whose complement just consists of branch cuts ending at branch pts or $\infty$. (Let's call such a domain a "branch domain")

<u>example</u>: find a branch of $\sqrt{e^z+1}$, also compute $\frac{d}{dz}\sqrt{e^z+1}$ for this branch.

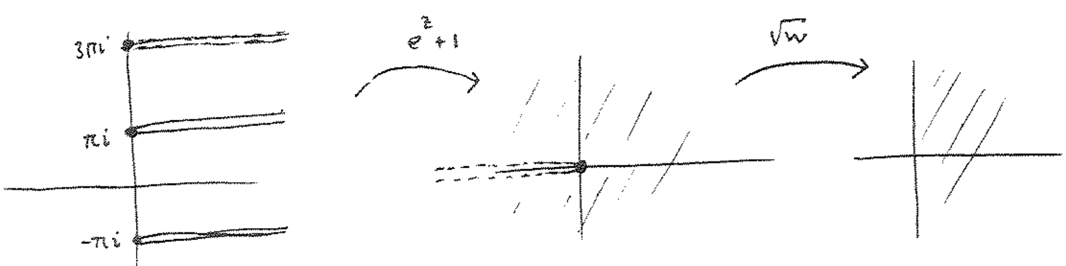
$f(z) = e^z + 1$

$g(w) = \sqrt{w}$

$g(f(z))$

method: identify branch points. (1) then decide (2) where to put the cuts, so that you construct a branch domain

one possible soln!



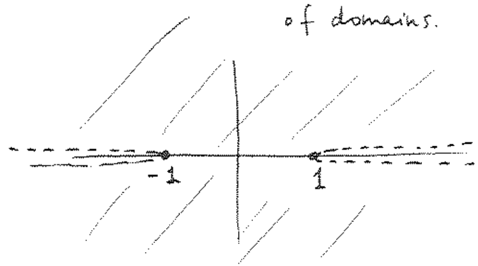
<u>example</u> find a branch domain and definition for

$$f(z) = \sqrt{z^2-1}$$

(1) write $f(z) = \sqrt{z-1}\sqrt{z+1}$


$0 < \arg(z-1) < 2\pi$ $\qquad$ $-\pi < \arg(z+1) < \pi$

so product is defined on intersection of domains.




so

$$f(z) = \sqrt{z-1}\sqrt{z+1} = \left(|z-1|\,|z+1|\right)^{1/2} e^{i\frac{1}{2}\left(\arg(z-1) + \arg(z+1)\right)}$$

$\arg(z-1) \in (0, 2\pi)$

$\arg(z+1) \in (-\pi, \pi)$

also can work:

(2) write $f(z) = g \circ h(z)$

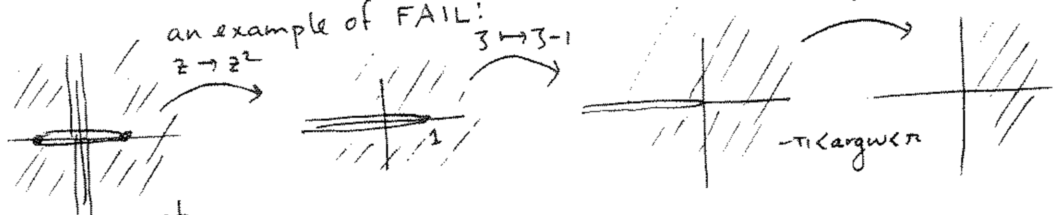
$h(z) = z^2 - 1$

$g(w) = \sqrt{w}$

started here

$\sqrt{w} = |w|^{1/2} e^{i\frac{\arg w}{2}}$

an example of FAIL:




not a connected domain → REJECT.

retry? same branch points, better cuts.