Math 4200-1
Monday 9/12

loose ends in §1.5

I forgot the punchline at the end of class on Friday
CR in polar coords is

$$r u_r = v_\theta$$
$$r u_\theta = -v_r \qquad \underset{\sim}{u} \quad \underset{\sim}{v}$$

for $\log z = \ln|z| + i \arg z = \ln r + i\theta$

$$r u_r = r\frac{1}{r} = 1 = v_\theta \quad \checkmark$$
$$r u_\theta = 0 = v_r$$

much easier than checking rectangular CR.

$f = u + iv$ analytic

CR $\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}$ if $\underset{v}{u} \in C^2$ then $\begin{array}{l} u_{xy} = u_{yx} \\ v_{xy} = v_{yx} \end{array}$

so $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$

C^2 (real-valued) functions $U(x,y)$ are called harmonic if $\underbrace{U_{xx} + U_{yy} = 0}$
this partial differential equation is called ~~the~~ Laplace Equation

(harmonic functions are important in pure & applied math)

Exercise Is $\mathrm{Im}(f(z)) = v(x,y)$ harmonic if f is analytic?

Def if $u \in C^2(A)$, A open, and if u is harmonic,
a function v so that
$f(x+iy) := u(x,y) + i v(x,y)$ is analytic is called a harmonic conjugate to u

<u>Theorem</u> If $u(x,y)$ is harmonic and C^2 in an <u>open</u> <u>simply connected</u> connected & domain (e.g. $D(z_0;r)$), then

roughly "no holes" we'll make this precise later

$\exists$ harmonic conjugate $v(x,y)$, unique up to an additive constant.

<u>pf</u>: $u(x,y) \in C^2$ is given. The system for $v(x,y)$ is

$$v_x = P(x,y) \qquad (= -u_y)$$
$$v_y = Q(x,y) \qquad (= u_x)$$

When you study <u>conservative vector fields</u> and <u>Green's Thm</u> in multivariable calc you learn that you can antidifferentiate to find v iff $P_y = Q_x$, which holds since $P_y = -u_{yy} = u_{xx} = Q_x$ since u is harmonic. ∎

<u>example</u> $u(x,y) = xy$

Show u is harmonic & find conjugate. Identify the analytic function $f(z) = u(x,y) + i v(x,y)$

<u>example</u> : $u(x,y) = \ln\sqrt{x^2+y^2}$ is harmonic in $\mathbb{C} \setminus \{0\}$. ($u = \mathrm{Re}(\log z)$). or check directly

Integrate CR in polar coords to illustrate why domains which are not simply connected may not have global harmonic conjugates

CR $\boxed{\begin{matrix} u_\theta = -r v_r \\ v_\theta = r u_r \end{matrix}}$

Fill in discussion of connectivity (simple connectivity comes later in course) from §1.4, and apply it to a thm related to last week's HW.

Def A is pathwise connected iff $\forall z_0, z_1 \in A$ $\exists$ cont. path $\gamma: [a,b] \to A$ with $\gamma(a) = z_0$, $\gamma(b) = z_1$

Def A is connected if whenever $A = \mathcal{O} \cup V$ with $\mathcal{O}, V$ open & $\mathcal{O} \cap V = \emptyset$, then either $\mathcal{O}$ or V is empty.

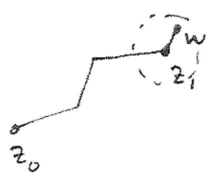
Theorem If $A \subset \mathbb{C}$ is open then A is connected iff A is pathwise connected. Furthermore in this case, $\forall z_0, z_1 \in A$ $\exists$ a piecewise C^1 path $\gamma = [\gamma_1, \gamma_2 .. \gamma_k]$ connecting z_0 to z_1. (Each γ_i can in fact be taken as a constant speed line segment parameterization.)

pf: Let A be connected (and open).

Pick $z_0 \in A$.

Let $B \subset A$, $B := \{ z \in A \text{ s.t. } \exists \text{ p.w. } C^1 \text{ path from } z_0 \text{ to } z \}$.

- B is open: if $z_1 \in B$, then $\exists r > 0$ s.t. $D(z_1; r) \subset A$ for any $w \in D(z_1; r)$ $\exists$ line-segment path from z_1 to w, which can be amalgamated with the p.w. C^1 path from z_0 to z_1 to give a p.w. C^1 path from z_0 to w.



- B is ~~open~~ closed: Let $z_1 \in \overline{B} \subset A$; $\Rightarrow \exists r > 0$ s.t. $D(z_1; r) \subset A$
  $\Rightarrow \exists w \in B \cap D(z_1; r)$.
  so $\exists$ p.w. C^1 path from z_0 to w amalgamate this with a line-segment path from w to z_1, so that $z_1 \in B$
  Thus $\overline{B} = B$ (in A).

$\Rightarrow B = A$ ∎

Now assume A is path connected.

If A is not connected $\exists$ B s.t. $B \subset A$, $B \neq \emptyset, A$, B is open and closed in A.

pick $z_0 \in B$, $z_1 \in A \setminus B$

and $\gamma : [a,b] \to A$ a continuous path. $\gamma(a) = z_0$, $\gamma(b) = z_1$

Let $t_1 = \sup \{ t \in [a,b] \text{ s.t. } \gamma(t) \in B \}$

If $\gamma(t_1) \in B$ then B open $\Rightarrow \exists r > 0$ s.t. $D(\gamma(t_1); r) \subset A$
γ cont $\Rightarrow \exists t > t_1$ s.t.
$\gamma(t) \in D(\gamma(t), r) \subset A$.
$\times$.

If $\gamma(t_1) \notin B$ then $A \setminus B$ open
$\Rightarrow \exists r > 0$ s.t. $D(\gamma(t_1); r) \cap B = \emptyset$
$\Rightarrow \nexists \{t_k\} \nearrow t_1$ s.t. $\gamma(t_k) \in B$. $\times$.

<u>Theorem</u> Let $A \subset \mathbb{C}$ be open and connected.

$f : A \to \mathbb{C}$ analytic, with $f'(z) = 0 \ \forall z \in A$.

then f is constant.

<u>proof</u>: A is pathwise connected, by previous theorem.
And any two points can be connected by a piecewise C^1 path.

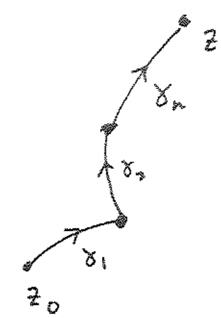
Fix any $z_0 \in A$.
We show $f(z) = f(z_0) \ \forall z \in A$.

<u>proof</u> Let $z_1 \in A$.

Let $\gamma = [\gamma_1, \gamma_2, \ldots \gamma_n]$ be a piecewise C^1 path connecting z_0 to z_1.

i.e. $\gamma_j : [a_j, b_j] \to \mathbb{C}$ is C^1

$\gamma_j(b_j) = \gamma_{j+1}(a_j) \quad j = 1, 2, \ldots n-1.$



Then $$f(z_1) - f(z_0) = \sum_{j=1}^{n} \int_{a_j}^{b_j} \underbrace{\frac{d}{dt} f(\gamma_j(t))\, dt}_{\underbrace{f'(\gamma_j(t))}\ \gamma_j'(t)\, dt}$$

chain rule for curves

complex deriv $\equiv 0$ by hypothesis.

$$= 0$$

∎

<u>Remark</u> The real variables version of this is analogous, and uses the real variables chain rule.

Thus if $A \subset \mathbb{R}^n$ is open and connected, $v : A \to \mathbb{R}$, $v \in C^1(A)$ & $\nabla v \equiv 0$ on A, then v is constant

$$\ldots \quad v(\vec{x}_1) - v(\vec{x}_0) = \sum_{j=1}^{n} \int_{a_j}^{b_j} \underbrace{\frac{d}{dt} v(\gamma_j(t))\, dt}_{\nabla v(\gamma_j(t)) \cdot \gamma_j'(t)\, dt \ \equiv \ 0} = 0.$$

For example, this is why harmonic conjugates are unique up to additive constants