Math 4200
Friday 10/7
§2.4 Cauchy Integral Formula & consequences

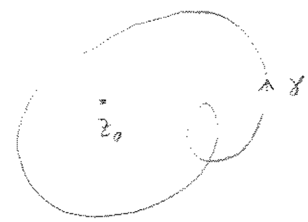
HW for 10/21

2.4 2, 3, 4, 7, 8, 12, 16, 17, 18
2.5 2, 5, 6, 8, 9a, 10, 15, 18
Class Exercise I (today's notes)

Recall, last Friday we proved C.I.F., for f analytic in A, γ homotopic to a pt. in A (as closed curves)

$$\boxed{f(z_0)\, I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\, dz}$$

$z_0 \notin \gamma(I)$



when $I(\gamma; z_0) \neq 0$ you can recover $f(z_0)$ from the values of f along γ

1st application: diffble $\Rightarrow$ ∞'ly diffble, with estimates for derivs which only depend on f:

rewrite C.I.F.

$$\boxed{f(z)\, I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta}$$

$z \notin \gamma(I)$

Theorem: f analytic in A $\Rightarrow$ f ∞'ly diffble in A. In fact for γ homotopic to a point in A,

$$\boxed{\begin{aligned} f'(z)\, I(\gamma; z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \\ f^{(n)}(z)\, I(\gamma; z) &= \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \end{aligned}}$$

notice, this is what we get by "differentiating thru the integral sign"

$$\frac{d}{dz}\frac{f(\zeta)}{\zeta - z} = f(\zeta)(-1)(\zeta - z)^{-2}(-1) = \frac{f(\zeta)}{(\zeta - z)^2}$$

$$\frac{d}{dz}\frac{f(\zeta)}{(\zeta - z)^n} = f(\zeta)(-n)(\zeta - z)^{-n-1}(-1) = n\frac{f(\zeta)}{(\zeta - z)^{n+1}}$$

So, when can you justify this operation?
"That's an analysis question"

analysis answer: (to justify the differentiation)

let $G(z) = \int_\gamma g(z,\zeta)\,d\zeta$

$$\frac{G(z+h)-G(z)}{h} = \int_\gamma \frac{1}{h}\left(g(z+h,\zeta)-g(z,\zeta)\right)d\zeta$$

$$\overset{?}{\to} \int_\gamma \frac{\partial g}{\partial z}(z,\zeta)\,d\zeta \quad , \text{ as } h\to 0$$

certainly need: $g(z,\zeta)$ complex differentiable in z

then, following suffices: the difference quotients converge uniformly (wrt $\zeta \in \gamma(I)$) to $\frac{\partial g}{\partial z}(z,\zeta)$:

$$\boxed{\begin{array}{l}\forall \varepsilon>0\ \exists \delta>0 \text{ s.t.} \\ |h|<\delta \Rightarrow \left|\frac{g(z+h,\zeta)-g(z,\zeta)}{h} - \frac{\partial g}{\partial z}(z,\zeta)\right| < \varepsilon\end{array}}$$

If box holds, then

$$|h|<\delta \Rightarrow \left|\frac{G(z+h)-G(z)}{h} - \int_\gamma \frac{\partial g}{\partial z}(z,\zeta)\,d\zeta\right| \le \int_\gamma \left|\frac{g(z+h,\zeta)-g(z,\zeta)}{h} - \frac{\partial g}{\partial z}(z,\zeta)\right| |d\zeta|$$

$$< \varepsilon \underset{\substack{\uparrow \\ \text{length}}}{L}(\gamma) ,$$

which implies

$$G'(z) = \int_\gamma \frac{\partial g}{\partial z}(z,\zeta)\,d\zeta$$

so, how close is $\frac{g(z+h,\zeta)-g(z,\zeta)}{h}$ to $\frac{\partial g}{\partial z}(z,\zeta)$?

|| FTC

$$\frac{1}{h}\int_{z\to z+h} \frac{\partial}{\partial w} g(w,\zeta)\,dw = \underbrace{\frac{1}{h}\int_{z\to z+h} \frac{\partial}{\partial z} g(z,\zeta)}_{\frac{\partial g}{\partial z}(z,\zeta)} + \underbrace{\left[\frac{\partial g}{\partial w}(w,\zeta) - \frac{\partial g}{\partial z}(z,\zeta)\right] dw}_{}$$

$$\frac{\partial g}{\partial z}(z,\zeta) \quad + \quad \underbrace{\frac{1}{h}\int_{z\to z+h} \left[\frac{\partial g}{\partial w}(w,\zeta) - \frac{\partial g}{\partial z}(z,\zeta)\right] dw}_{\text{error term}}$$

If $\frac{\partial g}{\partial w}(w,\zeta)$ is continuous on $\underset{\substack{\downarrow \\ w}}{\overline{D(z,\rho)}} \times \underset{\substack{\downarrow \\ \zeta}}{\gamma([a,b])}$ then it is uniformly continuous, so

Is $|\cdot| < \varepsilon$ uniformly along γ, for $|h|<\delta$?

$\forall \varepsilon>0\ \exists\ \delta$ s.t. $|h|<\delta \Rightarrow \left|\frac{\partial g}{\partial w}(w,\zeta) - \frac{\partial g}{\partial z}(z,\zeta)\right| < \varepsilon$

$\forall\ w \in D(z,\delta) \Rightarrow |\text{error term}| \le \frac{|h|}{|h|}\varepsilon$

$\forall\ \zeta \in \gamma([a,b])$

In our case

$$g(z,\zeta) = \frac{f(\zeta)}{(\zeta - z)^n}$$

$$\frac{\partial g}{\partial w}(w,\zeta) = \frac{n f(\zeta)}{(\zeta - w)^{n+1}}$$ is continuous on $\overline{D(z;\rho)} \times \gamma([a,b])$

as soon as ρ small enough that $\overline{D(z,\rho)} \cap \gamma([a,b]) = \emptyset$.

∎

Two beautiful consequences of the differentiation theorem:

<u>Liouville's Theorem</u>: Let $f: \mathbb{C} \to \mathbb{C}$ be <u>entire</u> ($\mathbb{C}$-diff'ble $\forall z \in \mathbb{C}$) and <u>bounded</u>. ($\exists M$ s.t. $|f(z)| \leq M$ $\forall z \in \mathbb{C}$)

Then f is constant!

<u>proof</u>: Let $z \in \mathbb{C}$, $R > 0$, $\gamma_R(t) = z + Re^{it}$ $0 \leq t \leq 2\pi$

so $I(\gamma; z) = 1$.

Thus
$$f'(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

$$\Rightarrow |f'(z)| \leq \frac{1}{2\pi} \int | \quad | |d\zeta| \leq \frac{1}{2\pi} \frac{M}{R^2} 2\pi R = \frac{M}{R}$$

true $\forall R > 0$. Let $R \to \infty \Rightarrow |f'(z)| = 0$ $\forall z$

$\Rightarrow f$ constant ∎

<u>Fundamental Theorem of Algebra</u>:

Let $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a poly of degree n (scaled so that coeff of z^n is 1).

Then $p(z)$ factors into a product of linear factors,

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_n), \quad z_i \in \mathbb{C}.$$

<u>proof</u>:

- it suffices to prove $\exists$ <u>a</u> linear factor when $n \geq 1$, since general case then follows by induction:
  - I) FTA true when $n = 1$
  - II) If true for $n-1$, and $p_n(z) = (z - z_n) p_{n-1}(z)$ then true for $p_n(z)$.

- it suffices to prove $\exists$ a root when $n \geq 1$, since if $p_n(z_n) = 0$
  then $z - z_n$ is a factor of $p_n(z)$:

$$\frac{p_n(z)}{z - z_n} = q_{n-1}(z) + \frac{R}{z - z_n}$$ from division algorithm.

$$\Rightarrow p_n(z) = q_{n-1}(z)(z - z_n) + R$$

$$\Rightarrow p_n(z_n) = 0 + R$$ in general; if $p_n(z_n) = 0$ then $R = 0$.

So we prove that (for $n \geq 1$) $p_n(z)$ has a root.
Proof is by contradiction.

If $p(z) = p_n(z)$ does <u>not</u> have a root
then $\frac{1}{p(z)}$ is entire! Now we'll show it's bounded, $\exists M$ s.t. $\left|\frac{1}{p(z)}\right| \leq M$ $\forall z$.
This implies $\frac{1}{p(z)} = \widetilde{\text{const}}$ by Liouville, i.e. $p(z) = \text{const}$ $\times$.

- $\lim_{|z| \to \infty} \left|\frac{1}{p(z)}\right| = 0$.

pf: $p(z) = z^n\left(1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right)$

$$|p(z)| \geq |z|^n\left(1 - \left|\frac{a_{n-1}}{z}\right| - \left|\frac{a_{n-2}}{z^2}\right| - \dots - \left|\frac{a_0}{z^n}\right|\right)$$

Let $A = \max_{0 \leq i \leq n-1} |a_{n-i}|$

Then $|z| \geq \max(1, 2nA) \Rightarrow \left|\frac{a_{n-j}}{z^j}\right| \leq \frac{|a_{n-j}|}{|z|} \leq \frac{A}{2nA} = \frac{1}{2n}$

$$\Rightarrow |p(z)| \geq |z|^n\left(\tfrac{1}{2}\right)$$

$$\Rightarrow \frac{1}{|p(z)|} \leq \frac{2}{|z|^n}$$ ∎

- Letting $R = \max(1, 2nA)$
  We see for $|z| \geq R$, $\left|\frac{1}{p(z)}\right| \leq \frac{2}{R^n}$
  for $|z| \leq R$, $\left|\frac{1}{p(z)}\right| \leq M_1$ because $\left|\frac{1}{p(z)}\right|$ is continuous on this compact set.

Thus for $M = \max\left(M_1, \frac{2}{R^n}\right)$

$$\left|\frac{1}{p(z)}\right| \leq M \quad \forall z \in \mathbb{C}.$$

$\times$, see above!

Another consequence:

<u>Morera's Theorem</u> : Let $f: A \to \mathbb{C}$ continuous

<u>and</u> satisfy rectangle condition

$$\oint_{\partial R} f(z)\,dz = 0 \qquad \forall R \subset A \text{ rectangle.}$$

then $f(z)$ is analytic.

<u>proof</u> : These hypotheses guarantee $f(z)$ has a local complex antiderivative $F(z)$; $F'(z) = f(z)$

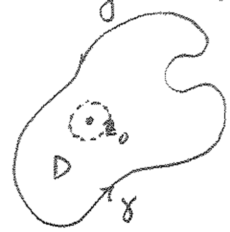
(see §2.3 notes, sept [illegible])

But by the Cauchy derivative formulas, $F(z)$ is infintely complex differentiable.
In particular, $F''(z) = f'(z)$ exists!

Class exercise I:

Recall that if γ is p.w. C^1 boundary ∂D of a bounded domain D, and if $z_0 \in D$, then

$$I(\gamma; z_0) = 1$$



So in this case the Cauchy integral formula reads

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz \qquad \text{(for analytic } f\text{)}$$

Assuming f is not only analytic but also C^1 in an open domain containing $\overline{D}$ this special case of C.I.F. can be proven with Green's Theorem. Here's how: By the contour replacement theorem,

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = \oint_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz$$

where r is small enough so that $D(z_0; r) \subset D$.

approximate $f(z)$ by $f(z_0)$ + error in the integral on the right, and rigorously compute $\lim_{r\to 0}$.

Carry out this argument.