Math 4200
Mond. 10/31.

§3.2–3.3.

On Friday we showed that every analytic function $f(z)$ has a local power series representation that converges on the largest $D(z_0; r)$ s.t. $f(z)$ can be defined (or redefined) as an analytic function.

One consequence of power series representation for analytic functions is the isolated zeroes theorem, which has the surprising consequence that if two analytic functions agree on a convergent sequence $\{z_k\}$ with $\{z_k\} \to z_0 \in A$ and $z_k \neq z_0 \ \forall k$, then they agree everywhere on their common connected domain.

Thus if $f: A \to \mathbb{C}$ is analytic, with A open & connected, then any extension of f to to a large connected domain $\tilde{A}$, $A \subset \tilde{A}$, $\tilde{f}: \tilde{A} \to \mathbb{C}$ analytic, $f = \tilde{f}$ on A

is unique.

By a stroke of good fortune, this week's undergraduate colloquium, by Aaron Wood,

Wed LCB 225
12:55–1:45

will explain why

$$1 + 2 + 3 + 4 + \dots = -\frac{1}{12}$$

makes sense.

(This is $\zeta(-1)$, where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann Zeta function, or rather its analytic extension which, unlike the sum, is valid for $Re(s) \leq 1$.)

http://www.math.utah.edu/ugrad/colloquia.html

Sometimes its useful to know you can multiply Power series term by term

<u>Theorem</u> If $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$

$$g(z)=\sum_{n=0}^{\infty} b_n(z-z_0)^n$$

in $D(z_0;R)$, then

$$f(z)g(z) = a_0b_0 + (a_1b_0+a_0b_1)(z-z_0) + (a_2b_0+a_1b_1+a_0b_2)(z-z_0)^2 + \cdots$$

$$= \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m} a_j b_{m-j}\right)(z-z_0)^m \qquad \text{in } D(z_0;R)$$

slick proof: $f(z)g(z)$ is analytic in $D(z_0;R)$, so has a power series

$$f(z)g(z) = \sum_{m=0}^{\infty} \frac{(fg)^{(m)}(z_0)}{m!}(z-z_0)^m$$

but $(fg)(z_0) = a_0b_0$

$(fg)'(z_0) = (f'g+fg')(z_0) = a_1b_0 + a_0b_1$

by induction,

$$(fg)^{(m)} = f^{(m)}g + m f^{(m-1)}g' + \binom{m}{2} f^{(m-2)}g^{(2)} + \cdots + \binom{m}{m-1} f' g^{(m-1)} + fg^{(m)}$$

@ z_0, $(fg)^{(m)}(z_0) = m!\,a_m b_0 + m\cdot(m-1)!\,a_{m-1}b_1$

$$+ \cdots + \frac{m!}{(m-k)!\,k!}(m-k)!\,a_{m-k}(k!)b_k + \cdots$$

$$= m!\left(\sum_{j=0}^{m} a_{m-j}b_j\right) = m!\left(\sum_{j=0}^{\infty} a_j b_{m-j}\right) \qquad \blacksquare$$

<u>Exercise 1</u>: Find the first few terms in Taylor series for $\tan z = \frac{\sin z}{\cos z}$ @ $z_0=0$

<u>hint</u>: rewrite as $(\cos z)(\tan z) = \sin z$.

1b) radius of conv?

§ 3.3 Laurent Series

These are power series in positive and negative powers, valid in annuli.

We'll prove that

f is analytic in an annulus $D(z_0;R) \setminus \mathrm{cl}(D(z_0,r))$ [or in a punctured disk at z_0]

$= \{z \in \mathbb{C} \text{ s.t. } \cancel{\ } r < |z-z_0| < R\}$

iff

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\substack{\text{convergent for } |z-z_0|<R \\ (\text{unif abs conv for } |z-z_0| \le R-\varepsilon)}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\substack{\text{convergent for } |z-z_0|>r \\ (\text{unif abs conv for } |z-z_0| \ge r+\varepsilon)}}.$$

<u>Exercise 2</u> Let $f(z) = \dfrac{1}{(z-1)(z+2)}$

Find, using geometric series, and partial fractions

a) Taylor series @ $z_0 = 0$

b) Laurent series for $1 < |z| < 2$

c) Laurent series for $|z| > 2$

We'll finish proving this on Wed.... but ② ⇒ ① is quick.

<u>Laurent series</u> : <u>Theorem</u>

Let $A = \{ z \mid r_1 < |z - z_0| < r_2 \}$ be an open annulus. Then

① $f : A \to \mathbb{C}$ analytic,

$\boxed{\text{iff}}$

② $f(z)$ has a power series expansion (with positive & negative powers),

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m}$$

$$:= S_1(z) + S_2(z)$$

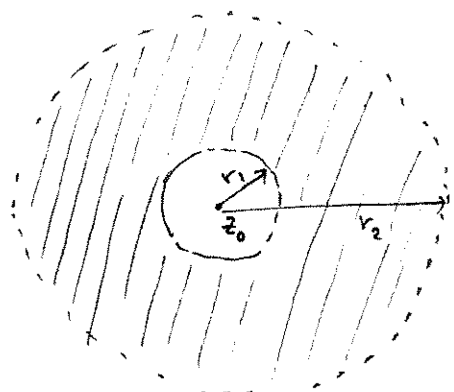
Here $S_1(z)$ converges for $|z - z_0| < r_2$ (uniformly absolutely for $|z - z_0| \leq r_2 - \varepsilon$, $\varepsilon > 0$)

$S_2(z)$ converges for $|z - z_0| > r_1$ (uniformly absolutely for $|z - z_0| \geq r_1 + \varepsilon$, $\varepsilon > 0$)

) Furthermore, the coefficients a_n, b_m are uniquely determined by f; specifically, if $r_1 < r < r_2$, and if $\gamma = \{ z \mid |z - z_0| = r, \ r_1 < r < r_2 \}$ is the radius r circle centered at z_0, then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$$

$$b_m = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\zeta - z_0)^{m-1} \, d\zeta$$



② ⇒ ① : Since $S_1(z)$ and $S_2(z)$ converge uniformly on the sub-annuli $\{ z \mid r_1 + \varepsilon < |z - z_0| < r_2 - \varepsilon \}$ ~~their limits are~~ analytic, and so $S_1(z) + S_2(z)$ is as well. ■

Note too that $S_1(z)$ converges for $|z - z_0| < r_2$ <u>implies</u> uniform absolute convergence for $|z - z_0| \leq r_2 - \varepsilon$: (We already had this discussion for power series.....

... conv. for $|z - z_0| = r_2 - \varepsilon/2 \Rightarrow M = \max \{ |a_n| (r_2 - \varepsilon/2)^n, \ n = 0, 1, 2 .. \} < \infty$

then $|z - z_0| \leq r_2 - \varepsilon \Rightarrow |a_n (z - z_0)^n| \leq |a_n| (r_2 - \varepsilon)^n \leq M \left(\frac{r_2 - \varepsilon}{r_2 - \varepsilon/2} \right)^n = M \mu^n$, $\mu < 1$

$$\sum_{n=0}^{\infty} M \mu^n = \frac{M}{1 - \mu} < \infty \quad ■$$

Similarly, $S_2(z)$ converges for $|z - z_0| > r_1$ implies uniform absolute convergence for $|z - z_0| \geq r_1 + \varepsilon$. See HW.