Math 4200
Fri 10/28

HW for Fri 11/4
3.2 5c, 7, 13, 14, 18, 19, 20
3.3 1ab, 4, 6, 8, 9, 13, 15, 17, 18, 19, 20

§3.2 Taylor series convergence for analytic functions

Review:
We know

- limits of analytic fcns are analytic
  (if conv. is uniform on closed subdisks)
- the limit of the deriv. fcns is the deriv. of the limit fcn
- Weierstrass-M test for checking uniform absolute, hence uniform convergence of series of fcns
- Power series
  radius of convergence $R = \sup\{ r \geq 0 \text{ s.t. } \sum_{n=1}^{\infty} |a_n| r^n < \infty \}$
  derivative $\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n \right)' = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n} \qquad |z-z_0| < R.$

  So if $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall\, |z-z_0| < R$
  then a_n must equal $\frac{f^{(n)}(z_0)}{n!}$, i.e. uniqueness of a_n.

new

If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ has radius of convergence $R > 0$

then each antiderivative $F(z)$ of f in $D(z_0; R)$ is given by power series
$$F(z) = F(z_0) + \sum_{n=0}^{\infty} a_n \frac{(z-z_0)^{n+1}}{n+1}.$$

pf: the radius of conv. for F is at least R, since $\sum \frac{|a_n|}{n+1} r^{n+1} = r \sum \frac{|a_n|}{n+1} r^n \leq r \sum |a_n| r^n$
and $F' = f$ by diff thm.
(radius of conv for F can't be $> R$, since in that case radius for f would also be $> R$). ∎

So, power series ⇒ analytic
Today: analytic ⇒ power series
& consequences

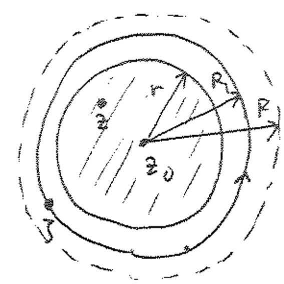
Theorem E : If f is analytic in $D(z_0;R)$ then its Taylor series converges:

let $a_n := \dfrac{f^{(n)}(z_0)}{n!}$

Then $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall z \in D(z_0;R)$

proof: let $|z-z_0| \le r < R_1 < R$

$\gamma = \{ \zeta \text{ s.t. } |\zeta - z_0| = R_1 \}$
c.c.



C.I.F. $\Rightarrow$ $f(z) = \dfrac{1}{2\pi i} \displaystyle\oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$= \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z_0} \left(\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}} \right) d\zeta \qquad \left| \frac{z-z_0}{\zeta-z_0} \right| \le \frac{r}{R_1} < 1$$

$$\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{\zeta-z_0} \right)^n$$

geometric series. uniform conv. for $|z-z_0| \le r$

$$= \frac{1}{2\pi i} \oint_\gamma \sum_{n=0}^{\infty} \underbrace{\frac{f(\zeta)}{(\zeta-z_0)^{n+1}} (z-z_0)^n}\, d\zeta$$

converges uniformly on γ : $\left| \dfrac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^{n+1}} \right| \le \dfrac{M}{R_1} \left(\dfrac{r}{R_1} \right)^n$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_\gamma \frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^{n+1}}\, d\zeta$$

interchange $\int$ & Σ ;
justified by uniform conv.

$$= \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\frac{1}{2\pi i} \oint \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta}_{\frac{1}{n!} f^{(n)}(z_0)}$$

by differention formula, from C.I.F !

$\square$

(notice, this implies the radius of convergence for the Taylor series of f is at least the largest R s.t. f is analytic in $D(z_0;R)$)

Theorem F : Isolated zeroes for analytic functions

Let $A \subset \mathbb{C}$ connected, open

$z_0 \in A$, $D(z_0; r) \subset A$

$f(z_0) = 0$

Then either $f(z) \equiv 0 \quad \forall z \in D(z_0; r)$

or $\exists \, \varepsilon > 0$ s.t. $f(z) \neq 0 \quad \forall \, 0 < |z - z_0| < \varepsilon$.

pf $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad |z - z_0| < r$

- if all $a_n = 0$ then $f \equiv 0$ in $D(z_0; r)$.
- else

$$f(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n \qquad a_N \neq 0$$

$$= (z - z_0)^N \sum_{n=N}^{\infty} a_n (z - z_0)^{n-N}$$

$$= (z - z_0)^N \underbrace{g(z)}$$

$g(z_0) = a_N^{\neq 0}$, g analytic $\Rightarrow$ cont

$\Rightarrow \exists \, \varepsilon > 0$ s.t. $|g(z)| > 0 \quad \forall \, |z - z_0| < \varepsilon$

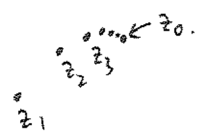
$\Rightarrow f(z) \neq 0 \quad \forall \, 0 < |z - z_0| < \varepsilon$.

This theorem has a somewhat amazing consequence:

Cor. Let A open & connected

$f, g : A \to \mathbb{C}$ analytic.

Suppose $f(z_k) = g(z_k) \quad \forall k \in \mathbb{N}$, for a sequence $\{z_k\} \to z_0 \in A$, with $z_0 \neq z_k \; k \in \mathbb{N}$.



Then $f = g$ on all of A.

pf: $f - g : A \to \mathbb{C}$ is analytic. $(f-g)(z_0) = \lim_{k \to \infty} (f-g)(z_k) = 0$.

By Theorem F, for $D(z_0; r) \subset A$,

$f - g \equiv 0 \quad \forall z \in D(z_0; r)$. (This is already surprising).

Now consider $B := \{ z \in A \text{ s.t. } (f-g)^{(n)}(z) = 0 \; \forall \, n = 0, 1, 2, \dots \}$.

We have

$D(z_0; r) \subset B$ since $f - g \equiv 0$ in $D(z_0; r)$

- B is closed in A: (if $\{w_k\} \to w \in A$, $(f-g)^{(n)}(w_k) \to (f-g)^{(n)}(w)$)
- B is open in A: if $z_1 \in B$, $D(z_1; r) \subset A \Rightarrow D(z_1; r) \subset B$

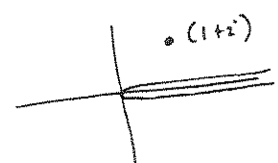
so $B = A$

<u>Example</u> Let $f(z)$ be entire.
Is it possible for $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ $\forall n \in \mathbb{N}$ and $f(-1) = -1$?

<u>Example</u> Explain why the radius of convergence for the Taylor series of $\frac{1}{z^2 - z - 6}$ @ $z = 0$ is $R = 2$.

Check ans. by finding the Taylor series

<u>Example</u> What is the radius of convergence for the Taylor series for $\log z$, at $z_0 = 1 + i$
if $\log z := \ln|z| + i \arg z$
$0 \leq \arg z < 2\pi$



Check ans