Math 4200
Wed 10/26

3.1–3.2 cont'd.

We used Morera's Thm to prove

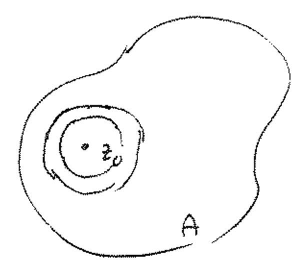
Theorem A Let $f_n : A \to \mathbb{C}$ analytic (A open), $n \in \mathbb{N}$.
If $f_n \to f$ uniformly in A (implied by $\{f_n\}$ uniformly Cauchy)
then $f : A \to \mathbb{C}$ is analytic.

Let's improve this result, to also include the fact that $f' = \lim f_n'$: (certainly not true in the $f: \mathbb{R} \to \mathbb{R}$ case!).

Theorem B Let $A \subset \mathbb{C}$ open
Let $f_n : A \to \mathbb{C}$ analytic, $n \in \mathbb{N}$.
Let $f_n(z) \to f(z)$ $\forall z \in A$, with the convergence uniform on each closed disk $cl(D(z_0, r)) \subset A$. (but not necessarily uniform in all of A)

- Then f is analytic.
- Furthermore, $f_n' \to f'$ $\forall z \in A$, and the convergence is uniform on each closed disk contained in A.

pf: 1st bullet point follows from Theorem A, applied to each $D(z_0; r)$ s.t. $cl(D(z_0; r)) \subset A$.
2nd bullet point:
Let $cl(D(z_0; r)) \subset A$
$\exists\, \varepsilon > 0$ s.t. $cl(D(z_0; r+\varepsilon)) \subset A$ as well



for $|z - z_0| \le r$,

$$f'(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r+\varepsilon} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

$$f_n'(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r+\varepsilon} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

by hypothesis, $f_n \to f$ uniformly on $\{ |\zeta - z_0| = r + \varepsilon \}$, so
also $\frac{f_n(\zeta)}{(\zeta - z)^2} \to \frac{f(\zeta)}{(\zeta - z)^2}$ uniformly.
on $|\zeta - z_0| = r + \varepsilon$
(since $|\zeta - z|^2 \ge \varepsilon^2$).

thus $$\int_{|\zeta - z_0| = r+\varepsilon} \frac{f_n(\zeta)}{(\zeta - z)^2}\, d\zeta \to \int_{|\zeta - z_0| = r+\varepsilon} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

we'll use Theorem B most in its series version

Theorem B′ $A \subset \mathbb{C}$ open

$g_n : A \to \mathbb{C}$ analytic

if $\sum_{n=1}^{\infty} g_n(z)$ converges uniformly on every closed disk in A, to $f(z)$

then $\left(\sum_{n=1}^{\infty} g_n(z)\right)' = \sum_{n=1}^{\infty} g_n'(z)$

and the series on the right converges uniformly on every closed disk

Exercise 1 On Monday we showed

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

converges for $|z|<1$ (uniformly for $|z| \le 1-\varepsilon$)

diverges for $|z| \geq 1$.

use the differentiation result (Theorem B′ above), to identify

$\sum_{n=0}^{\infty} n z^n$ in closed form

Exercise 2 On Monday we showed

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges $\forall z$ (uniformly for $|z| \le R$).

use the differentiation theorem to verify $f'(z) = f(z)$ $\forall z$.

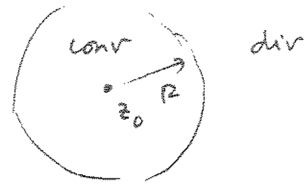
$f(0) = 1$

deduce $f(z) = e^z$

Power series

Consider the series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \begin{matrix} z_0 \in \mathbb{C} \\ a_n \in \mathbb{C} \end{matrix}$$



Then $\exists !\ R \in [0, \infty]$ s.t. this series converges $\forall z$ s.t. $|z - z_0| < R$,

uniformly for $|z - z_0| \le R - \varepsilon$,

and diverges $\forall\ |z - z_0| > R$ ($|z - z_0| = R$ is borderline).

Thus $f(z)$ is analytic for $|z - z_0| < R$,

and $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$

<u>proof</u>: let $R := \sup \{ r \ge 0 \text{ s.t. } \sum_{n=0}^{\infty} |a_n| r^n < \infty \}$

(a) If $|z - z_0| \le R - \varepsilon$ $(\varepsilon > 0)$

then $\sum_{n=0}^{\infty} |a_n| |z - z_0|^n \le \sum_{n=0}^{\infty} |a_n| (R - \varepsilon)^n < \infty$; so by Weirestrasse M-test $\sum a_n (z - z_0)^n$ converges uniformly for $|z - z_0| \le R - \varepsilon$

(b) If $|z_1 - z_0| > R$, write $|z_1 - z_0| = R_1$

we show $\sum a_n (z_1 - z_0)^n$ cannot converge.

if it did, then $\{a_n (z_1 - z_0)^n\} \to 0$, so $\max_n |a_n (z_1 - z_0)|^n \le C$.

so $|a_n| \le \dfrac{C}{R_1^n}$

and for $r < R_1$, $\sum |a_n| r^n \le \sum \dfrac{C}{R_1^n} r^n = C \sum \left(\dfrac{r}{R_1}\right)^n$ <u>converges</u>

$\Rightarrow R \ge R_1$ $\times$

<u>Theorem</u>: let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ have positive radius of convergence

<u>then</u> $a_n = \dfrac{f^{(n)}(z_0)}{n!}$, so in particular each coeff. a_n is uniquely determined.

<u>pf</u>: $f(0) = a_0$

$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$, $|z - z_0| < R$, $\Rightarrow f'(z_0) = 1 a_1$

$f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n (z - z_0)^{n-2}$ $\Rightarrow f''(z_0) = 2 a_2$

induct: $f^{(k)}(z) = \sum_{n=k}^{\infty} n! a_n (z - z_0)^{n-k}$ $\Rightarrow f^{(k)}(z_0) = k! a_k$ ∎

Theorem (proof on Friday!)

Let $f(z)$ be analytic in $D(z_0; r)$

Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{converges to } f(z) \text{ in } D(z_0;r)$$

In particular, the radius of convergence for the Taylor series is the largest radius R s.t. $f(z)$ is analytic in $D(z_0;r)$ $\forall r<R$.

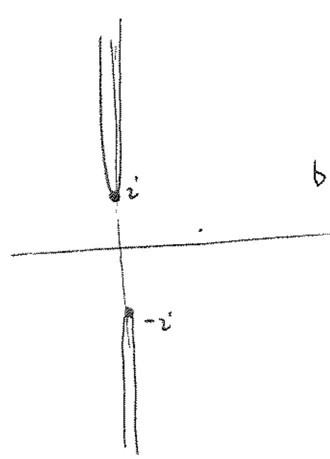
(Also, by uniqueness, if a power series converges to f, it is the Taylor series)

Exercise 3: What is the Taylor series for e^z, and what is its radius of convergence?

Exercise 4 Power series for $\log(1+z)$ @ $z_0=0$ must have $R=1$.
Find it without (or with Taylor)

Exercise 5 $\arctan z := \int_0^z \frac{1}{1+\zeta^2}\,d\zeta$

what is its Taylor series @ $z_0=0$?



branch domain for $\arctan z$