Math 4200
Mon 10/24

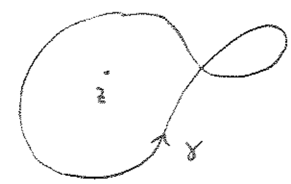
Chapter 3!!

Review: (since midterm exam)

Cauchy Integral formula

$$f(z)\,I(\gamma;z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

γ p.w. C' in A & closed
A open, simply connected.



Differentiation theorems:

$$f^{(n)}(z)\,I(\gamma;z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$

Estimates in case γ is a circle of radius R, centered at z:

$$|f^{(n)}(z)| \le \frac{n!}{R^n} M(R)$$

where $M(R) = \max\{|f(\zeta)| \text{ s.t. } |\zeta - z| = R\}$.

Corollaries

Liouville: If $f(z)$ is entire and bounded, f is constant (because $f'(z) = 0\ \forall z \in \mathbb{C}$)

HW: If $f(z)$ grows no faster than $|z|^n$, i.e.

$$|f(z)| \le C_1 + C_2|z|^n \quad \forall z$$

then $f(z)$ is a polynomial of degree $\le n$

since $|f^{(n+1)}(z)| \le \frac{(n+1)!}{R^{n+1}} M(R) \le \frac{(n+1)!}{R^{n+1}} \max\{|f(\zeta)|,\ |\zeta| \le |z| + R\}$

$\le \frac{(n+1)!}{R^{n+1}}\left(C_1 + C_2(|z| + R)^n\right)$

$\to 0$ as $R \to \infty$

Fund. thm of Algebra:
every $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$
factors into $(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$.

Morera's Theorem If $f: A \to \mathbb{C}$ is cont ($A \subset \mathbb{C}$ open)
and if rectangle condition holds:

$$\oint_{\partial R} f(z)\,dz = 0 \quad \forall \text{ closed rectangles } R \subset A$$

then f is analytic & C^∞ (pf in each $D(z_0;r) \subset A$ f will have an antideriv F; F is C^∞ by differentiation theorems, so $f = F'$ is too.

Cor: If $A \subset \mathbb{C}$ open & if $f_n: A \to \mathbb{C}$ are analytic, with $\{f_n\} \to f$ uniformly, then f is analytic

since $0 = \oint_{\partial R} f_n(z)\,dz \to \oint_{\partial R} f(z)\,dz$ (Morera).

uniform limits of analytic functions are analytic

Key for Chapter 3

Mean value property for f analytic

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta$$

for u harmonic

$$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\,d\theta$$

(or in complex notation

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})\,d\theta \quad)$$

Maximum ~~modulus~~ principle

$A \subset \mathbb{C}$ open, bounded, connected

$f: A \to \mathbb{C}$ analytic
$f: cl(A) \to \mathbb{C}$ continuous

then $|f(z)|$ maximum occurs on ∂A
if it also occurs at an interior point $z_0 \in A$, then f is constant

or

$u: A \to \mathbb{R}$ harmonic ($\& \, C^2$)
$u: cl(A) \to \mathbb{R}$ continuous

then maximum of $u(z)$ occurs on ∂A.
if it occurs at interior point then u is constant

We used the Maximum modulus principle to prove that the only analytic functions
$g: cl(D(0;1)) \to cl(D(0;1))$
$g: \{|z|=1\} \to \{|z|=1\}$
$g^{-1}: cl(D(0;1)) \to cl(D(0;1))$ exists and is analytic

are given by $f(z) = e^{i\theta}\dfrac{z - z_0}{1 - \bar{z}_0 z}$ for some $\theta \in [0,2\pi)$ and $z_0 \in D(0;1)$

We did not finish showing how these Möbius transformations can be used to transform the mean value property for harmonic functions into the Poisson integral formula for the unit disk

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{|e^{i\theta} - z_0|^2}\, u(e^{i\theta})\,d\theta.$$

(Let's postpone this derivation, in order to begin Chapter 3... the idea is to use the Möbius transformation to do a change of variables.)

§3.1 Sequences and series of analytic functions

Recall a key analysis theorem, which we mentioned last week.

Theorem 1 $A \subset \mathbb{R}^k$

$F_n : A \to \mathbb{R}^\ell$ continuous, $n = 1, 2, \ldots$

If $\{F_n\} \to F$ uniformly on A — i.e. $\forall \varepsilon > 0\ \exists N$ s.t. $n \geq N \Rightarrow |F(x) - F_n(x)| < \varepsilon\ \underline{\forall x \in A}$.

then F is continuous

We should go through the proof carefully, it's an important one.

Let $x_0 \in A$. Show the limit function F is continuous at x_0

(by using an F_n which is uniformly close to it)

↑ depending on ε

Corollary If $\{F_n\}$ is uniformly Cauchy (i.e. $\forall \varepsilon > 0\ \exists N$ s.t. $m, n \geq N \Rightarrow |F_n(x) - F_m(x)| < \varepsilon\ \forall x \in A$)

then $\exists$ a continuous limit function F

with $F_n \to F$ uniformly.

pf each sequence of points $\{F_n(x)\}$ has a limit since $\mathbb{R}^\ell$ is complete, call the limit point $F(x)$. Then $\{F_n(x)\} \to F(x)$ uniformly.

Corollary If $A \subset \mathbb{C}$ is open,

$f_n : A \to \mathbb{C}$ analytic $n = 1, 2, \ldots$

$\{f_n\}$ uniformly Cauchy of A, then $f_n(z) \to f(z)$ uniformly, and $f : A \to \mathbb{C}$ is analytic

Recall there is a 1-1 correspondence between

sequences $\{f_n(z)\}$ and series $\sum_{j=1}^{\infty} g_j(z)$

via the partial sums $\sum_{j=1}^{n} g_j(z)$:

Def: $\sum_{j=1}^{\infty} g_j(z)$ converges iff the sequence $\{f_n(z)\}$ does, with $f_n(z) := \sum_{j=1}^{n} g_j(z)$

and

each sequence $\{f_n(z)\}$ can be expressed as a series $\sum_{j=1}^{\infty} g_j(z)$,

with $g_1(z) := f_1(z)$

$g_2(z) := f_2(z) - f_1(z)$

$\vdots$

$g_n(z) := f_n(z) - f_{n-1}(z)$.

maybe M stands for modulus

Weierstrass M test for uniform (and absolute) convergence of series

Consider $\{g_n(z)\}$, $g_n : A \longrightarrow \mathbb{C}$ ($A \subset \mathbb{C}$ open as always)

if $\forall n \in \mathbb{N}$ $\exists$ M_n s.t. $|g_n(z)| \leq M_n$ $\forall z \in A$

and s.t. $\sum_{n=1}^{\infty} M_n < \infty$, then

$S_n(z) := \sum_{j=1}^{n} g_j(z)$ converges uniformly to a limit function on A

In fact, $\underbrace{\sum_{j=1}^{n} |g_j(z)| \text{ converge uniformly}}_{\text{this is called absolute convergence of the series.}}$ to a function $\leq \sum_{j=1}^{\infty} M_n$.

proof: We show $\{S_n\}$ is uniformly Cauchy.

Since $\underset{(\ell > k)}{|S_\ell(z) - S_k(z)|} = \left|\sum_{j=k+1}^{\ell} g_j(z)\right| \leq \sum_{j=k+1}^{\ell} |g_j(z)| \leq \sum_{j=k+1}^{\ell} M_j$ the results follow

(Let $\varepsilon > 0$. Pick N s.t. $k, \ell \geq N \Rightarrow \sum_{j=k+1}^{\ell} M_j < \varepsilon$. This proves the results).

Examples We discussed the zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\mathrm{Re}\, s > 1$, previously.

Try these:

$\sum_{n=0}^{\infty} z^n$ for $|z| < 1$ (~~do~~ consider $|z| < 1 - \varepsilon$)

$\sum_{n=1}^{\infty} n z^n$ for $|z| < 1$

$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ $\forall z$ (consider $|z| < R$)

hint: ratio test is good.
question: what analytic functions do you expect these series to converge to?