Math 4200
Fri 10/21

HW for Fri 10/28
Class exercise I in today's notes
2.5 #15, 18 (held over from this week)
3.1 #4, 6, 7, 12, 13, 14
3.2 2, 3, 4.

§2.5 cont'd

Recall, we proved the maximum modulus principle:

if $A \subset \mathbb{C}$ is open, connected, bounded
$f: A \to \mathbb{C}$ analytic
$f: \bar{A} \to \mathbb{C}$ continuous

then $\max_{z \in \bar{A}} |f(z)| = \max_{z \in \partial A} |f(z)|$

and if the maximum modulus occurs in A, then f is a constant function.

key tool: mean value property, which allowed us to prove that for $M := \max_{z \in \bar{A}} |f(z)|$,
then $B = \{z \in A \text{ s.t. } |f(z)| = M\}$
is open.

at the end of class I said you could use same technique to get max/min principle for harmonic fcns, which is true.
but, there's a slicker way to do it, as well as to prove the harmonic conjugate theorem, which we've been assuming from multivariable calc.

Theorem Let $A \subset \mathbb{C}$ be open and simply connected
$u: A \to \mathbb{R}$ harmonic and C^2.
Then $\exists$ harmonic conjugate $v: A \to \mathbb{R}^2$, and u, v are both C^∞
(so that $f = u + iv$ is analytic)

pf: if f existed then
$f' = u_x + iv_x$
$= v_y - iu_y$ any would be analytic

so consider $g(z) = u_x - iu_y$

CR for g: $(u_x)_x = (-u_y)_y$ ✓
$(-u_y)_x = -(u_x)_y$ ✓

So g is analytic,
A simply connected, so g has antideriv,
call it $f = U + iV$

then $f' = U_x + iV_x = u_x - iu_y$
$= V_y - iU_y$

$\Rightarrow U_x = u_x, \; U_y = u_y \Rightarrow U = u$ (after addition of a const) slick proof of max. modulus theorem for harmonic fcns:

$A \subset \mathbb{C}$ open, connected, bounded

$u: A \to \mathbb{R}$ harmonic, C^2

then $M := \max_{z \in \bar{A}} u(z) = \max_{z \in \partial A} u(z)$

and if M is attained at an interior point, u is constant

pf: as before, consider

$$B := \{ z \in A \text{ s.t. } u(z) = M \}$$

to show: B is open and closed ✓

$\Downarrow$

Let $z_0 \in B$. Let $D(z_0; r) \subset A$

$\exists$ harmonic conj $v \in D(z_0; r)$

s.t. $f = u + iv$ analytic

consider $e^{f(z)} = e^{u+iv}$

$|e^{f(z)}| = e^u = e^M$ @ z_0, and this is max value in $D(z_0; r)$

$\Rightarrow e^{f(z)} = \text{const}$ in $D(z_0; r)$

$\Rightarrow D(z_0; r) \subset B$ ∎

important application of maximum modulus theorem, which will also enable us to give a correct proof of the Poisson Integral formula, of a different flavor than the one in our text

<u>Question</u> : Consider $D(0;1) \subset \mathbb{C}$.

How many invertible conformal transformations $f : \overline{D(0,1)} \to \overline{D(0,1)}$
$\{|z|=1\} \to \{|w|=1\}$

are there? (i.e. f, f^{-1} are analytic bijections of the closed unit disk)

<u>Step 1</u> : What if we require $f(0) = 0$

consider $g(z) := \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(z) & z = 0 \end{cases}$

and $\frac{1}{g(z)}$

to deduce $f(z) = e^{i\theta} z$ is the only such conformal map.
(Use max. modulus principle!)

Step 2 Consider the Möbius transformation (see p 340 chapter 5)

a) $g(z) = \dfrac{z_0 + z}{1 + \bar{z}_0 z}$ $\quad z_0 \in D(0;1)$

show $|g(z)| = 1$ on $|z| = 1$, so by MMP, $|g(z)| < 1$ for $|z| < 1$

notice
$g(0) = z_0$

b) solve $w = \dfrac{z_0 + z}{1 + \bar{z}_0 z}$ for z to determine that $g^{-1}(w)$ is given by

$h(w) = \dfrac{w - z_0}{1 - w\bar{z}_0}$; verify $|h(w)| = 1$ on $|w| = 1$, so $|h(w)| < 1$ for $|w| < 1$.

thus g and h are examples of conformal bijections of $D(0;1)$

Step 3 Combine steps 1 & 2 to prove that every conformal diffeo of $D(0;1)$ is of the form $f(z) = e^{i\theta}\, g(z)$ for some $g(z)$ as above

($= e^{i\theta} h(z)$ for some $h(z)$)

these are the isometries of hyperbolic disk (another good project topic)

Poisson integral formula for unit disk: let u be harmonic (& C^2) in $D(0;1)$, and continuous on $\overline{D(0;1)}$.

$$\text{Then} \quad u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{|e^{i\theta}-z_0|^2}\, u(e^{i\theta})\, d\theta$$

* First: check why just using MVP for $f = u+iv$ only works at z_0

<u>trick</u>: we already know the mean value property. (Although it's slightly extended here, see class exercise)

consider $g(z) = \dfrac{z_0+z}{1+\bar{z}_0 z}$ from previous page

then $u\circ g$ is also harmonic on the disk & continuous on the closure.

note $u(z_0) = u\circ g(0)$!

$$u(g(0)) = \frac{1}{2\pi}\int_0^{2\pi} u(g(e^{i\alpha}))\, d\alpha = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, \frac{d\alpha}{d\theta}\, d\theta$$

change variables:

$g(e^{i\alpha}) = e^{i\theta}$; $h(e^{i\theta}) = e^{i\alpha}$

chain rule for curves

$$g'(e^{i\alpha})\, e^{i\alpha}\, \cancel{i}\, \frac{d\alpha}{d\theta} = \cancel{i} e^{i\theta}$$

$$\frac{d\alpha}{d\theta} = e^{i(\theta-\alpha)} \underbrace{\frac{1}{g'(e^{i\alpha})}}_{= h'(e^{i\theta})}$$

So

$$\boxed{u(z_0) = u(g(0)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{|\underset{\uparrow\, e^{i\theta}}{w}-z_0|^2}\, u(e^{i\theta})\, d\theta}$$

$$h(w) = \frac{w-z_0}{1-w\bar{z}_0}$$

$$h'(w) = \frac{(1-w\bar{z}_0) - (w-z_0)(-\bar{z}_0)}{(1-w\bar{z}_0)^2} = \frac{1-|z_0|^2}{(1-w\bar{z}_0)^2}$$

$$\frac{d\alpha}{d\theta} = \frac{e^{i\theta}}{\underset{h(e^{i\theta})}{e^{i\alpha}}}\, \frac{(1-|z_0|^2)}{(1-w\bar{z}_0)^2}$$

$$= e^{i\theta}\, \frac{(1-|z_0|^2)}{(1-w\bar{z}_0)^{\cancel{2}}} \left(\frac{\cancel{1-w\bar{z}_0}}{w-z_0}\right) = \frac{w(1-|z_0|^2)}{(1-w\bar{z}_0)(w-z_0)} \quad \frac{\bar{w}}{\bar{w}}$$

$$= \frac{(1-|z_0|^2)}{(1-w\bar{z}_0)(1-\bar{w}z_0)}$$

$$= \frac{1-|z_0|^2}{|1-w\bar{z}_0|^2} = \frac{1-|z_0|^2}{|w-z_0|^2}$$

!!

Class exercise I.

We know the mean value property for analytic functions f and harmonic functions u under the assumptions that f is analytic (resp. $u \in C^2$ and harmonic) on an open domain containing $cl(D(z_0; R))$:

$$f(z_0) = \frac{1}{2\pi} \int_{|z-z_0|=R} f(z_0 + re^{i\theta})\, d\theta$$

$$u(z_0) = \frac{1}{2\pi} \int_{|z-z_0|=R} u(z_0 + re^{i\theta})\, d\theta$$

Prove that these mean value properties hold under the weaker hypotheses

f analytic in $D(z_0; R)$ and continuous on $cl(D(z_0; R))$

u harmonic & C^2 in $D(z_0; R)$ and continuous on $cl(D(z_0; R))$.

Hint: the mean value properties hold for $r < R$, since the old hypotheses are valid for $D(z_0; r)$. Show that the mean value properties hold under the weaker hypotheses by letting $r \nearrow R$ and showing that the integrals on the circles $|z-z_0| = r$ converge to the ones on $|z-z_0| = R$.