Math 4200
Wed 10/19

§2.5 Maximum modulus principles for analytic and harmonic functions.

Recall on Monday we use C.I.F. to show that

<u>Theorem 1</u> $f: A \to \mathbb{C}$ analytic, $cl(D(z_0; R)) \subset A$

$$\Rightarrow \quad f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta$$

then, via conjugate function theory, we also deduced

closure, not conjugate

<u>Theorem 2</u> $u: A \subset \mathbb{R}^2 \to \mathbb{R}^2$ harmonic and C^2, $\overline{B_R(x_0,y_0)} \subset A$

$$\Rightarrow \quad u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\,d\theta$$

These mean value properties imply <u>maximum modulus principle</u> (for analytic fcns)
<u>maximum/minimum principles</u> (for harmonic fcns)

<u>Maximum modulus principle</u>

let $A \subset \mathbb{C}$ open, connected, bounded
$f: A \to \mathbb{C}$ analytic
$f: \overline{A} \to \mathbb{C}$ continuous.

Then $\max\{|f(z)| \text{ s.t. } z \in \overline{A}\} = \max\{|f(z)| \text{ s.t. } z \in \partial A\} := M$

i.e. the max of $|f(z)|$ occurs on ∂A

Furthermore if $\exists\, z_0 \in A$ (i.e. not on ∂A)
with $|f(z_0)| = M$, then f is a constant function of A.

Exercise 1 What is the maximum modulus of $f(z) = (z-2)^2$ on $cl(D(0;2))$?

proof of maximum modulus principle:

let $B = \{ z \in A \text{ s.t. } |f(z)| = M \}$

our goal is to show that either

(i) $B = \emptyset$, which implies that all points for which $|f(z)| = M$ satisfy $z \in \partial A$

OR

(ii) $B = A$. In this case $|f(z)|$ is constant, i.e.

for $f = u + iv$, $u^2 + v^2 \equiv M^2$. If $M = 0$ then $f = 0$

else $\cancel{2} u u_x + \cancel{2} v v_x \equiv 0$
$\cancel{2} u u_y + \cancel{2} v v_y \equiv 0$
$\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

since $M \neq 0$, $\begin{bmatrix} u \\ v \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

so $\det = 0$.

But $\det = u_x^2 + u_y^2 = v_y^2 + v_x^2$ (CR)

$\Rightarrow u, v$ are const ▨

OLD HW :)

$\Rightarrow f$ is const too

Since A is connected, it suffices to show that B is open and closed in A.

- Use mean value property to show B is open

(and f const on A, & f cont $\Rightarrow f$ const on $\bar{A}$)

- Use continuity to show B is closed. (in A)

<u>Exercise 2</u> Let $A \subset \mathbb{C}$ open and bounded.
Let $f, g : A \to \mathbb{C}$ analytic
$f, g : cl(A) \to \mathbb{C}$ continuous.

Show $\max_{z \in cl(A)} |f(z) - g(z)| \leq \max_{z \in \partial A} |f(z) - g(z)|$.

In particular, if $f = g$ on ∂A, then $f = g$ on $\bar{A}$

<u>Exercise 3</u> How would you prove the maximum/minimum principle for harmonic functions?

<u>Thm</u> Let A be a bounded open subset of $\mathbb{R}^2$, also connected
Let $u : A \to \mathbb{R}$ harmonic
$u : \bar{A} \to \mathbb{R}$ continuous.

Let $m = \min_{(x,y) \in \bar{A}} u(x,y)$; $M = \max_{(x,y) \in \bar{A}} u(x,y)$

Then the minimum m & the maximum M of u on $\bar{A}$ both occur on ∂A.
If $\exists\, (x_0, y_0) \in A$ with $u(x_0, y_0) = m$ or $u(x_0, y_0) = M$, then $u(x,y)$ is constant on $\bar{A}$.

There is an analog of the Cauchy Integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial A} \frac{f(z)}{z - z_0}\, dz$$

for harmonic functions, which lets you recover $u(x_0,y_0)$ from ~~its~~ the boundary values of u, as long as ∂A is piecewise C^1.
(note, the max/min principle shows that the boundary values of u on ∂A determine u in A).

For general domains this "Green's function representation" is complicated.
For disks we can derive it from the Cauchy Integral Formula.
The result is called the $\boxed{\text{Poisson Integral Formula}}$

<u>Theorem</u> Let $u : B_R(0) \to \mathbb{R}^2$ be harmonic and C^2 ($B_R(0) \subset \mathbb{R}^2$)
$u : \overline{B_R(0)} \to \mathbb{R}$ continuous

Then (using polar coords and abusing notation, for $\rho < R$

book typo page 173, but correct on page 175

$$u(\rho e^{i\phi}) = \frac{R^2-\rho^2}{2\pi} \int_0^{2\pi} \frac{u(Re^{i\theta})}{R^2+\rho^2-2R\rho\cos(\phi-\theta)}\, d\theta \quad \left(= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-\rho^2}{(R-\rho\cos(\phi-\theta))^2+\rho^2\sin^2(\phi-\theta)} f(\theta)\, d\theta \right)$$

expresses u inside the disk in terms of its boundary values... notice if $\rho = 0$, recover mean value property.

(Conversely, give a continuous function $f(\theta)$, if u define $u(Re^{i\theta}) := f(\theta)$, this formula extends f as (the unique) harmonic function inside the disk ... this extension thm is proven in PDE classes.)

(topics related to harmonic fcns can make for good projects.)

(partial proofs on Friday.).