Math 4200
Mon. 10/17.

Review our results from Friday before break.
These were consequences of C.I.F.

<u>Theorem 1</u> f analytic in A, γ homotopic to a pt. in A (as a closed curve), $z \in A$

$$\Rightarrow f^{(n)}(z)\, I(\gamma; z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

we used analysis to justify differentiating through the integral sign.

<u>Theorem 2</u> Liouville's Thm: If f is analytic on all of $\mathbb{C}$ ("entire") and if f is bounded, $|f(z)| \le M \ \forall z \in \mathbb{C}$, then f is constant.

pf: for $\gamma = z_0 + Re^{it}$ $0 \le t \le 2\pi$,

$$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2}\, d\zeta \quad \Rightarrow \quad |f'(z_0)| \le \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R}$$

$$R \to \infty \Rightarrow f'(z_0) = 0 \ \forall z_0 \in \mathbb{C}. \ \blacksquare$$

<u>Theorem 3</u> Let $p_n(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a poly of degree n. Then $p(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$. Fund. Thm. Alg.

pf: by induction find just 1 root.
if no root then $\frac{1}{p(z)}$ is entire and bounded ($|\frac{1}{p(z)}| \to 0$ as $z \to \infty$, and is bounded on any compact set.

$$\Rightarrow \frac{1}{p(z)} = C_1 \Rightarrow p(z) = \frac{1}{C_1} \ \text{※}. \ \blacksquare$$

(Later on we will see how certain contour integrals, $\int_\gamma \frac{f'(z)}{f(z)}\, dz$, count the # of zeroes of f inside of γ, and this will give an alternate constructive proof of FTA, as well as practical algorithms for finding the zeroes, approximately).

We did not get to

Theorem 4 Morera's Theorem.

Let $f: A \to \mathbb{C}$ be (just) continuous, s.t. the rectangle lemma holds, i.e. $\forall\, R = \{ z = x+iy,\ a \le x \le b,\ c \le y \le d \} \subset A$,

$$\oint_{\partial R} f(z)\,dz = 0.$$

Then f is analytic on A.

proof: Rectangle lemma $\Rightarrow$ f has an antideriv F in disks $D(z_0;R) \subset A$

$$\left(\text{recall}, \quad F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta = \int_{z_0}^{z} f(\zeta)\,d\zeta \right).$$

i.e. $F'(z) = f(z)\ \forall z \in D(z_0;R)$.

but F analytic $\Rightarrow$ $F', F'', \dots F^{(n)}$ exist $\forall n \in \mathbb{N}$ (Theorem 1)

in particular, $F''(z) = f'(z)$ exists $\forall z \in D(z_0;R)$ $\Rightarrow \forall z_0 \in A$.

Corollary: Let $\{f_n\}: A \to \mathbb{C}$ be a sequence of analytic functions. If $\{f_n\} \to f$ <u>uniformly</u> on A, then f is also analytic.

[contrast with real diff'ble case!].

pf: can you check these pieces, and combine them into a proof?

(i) f is continuous, because uniform limits of continuous fcns are cont.

(ii) $f_n \to f$ unif $\Rightarrow \oint_{\partial R} f_n\,dz \to \oint_{\partial R} f(z)\,dz$

so f satisfies the hypotheses of Morera.

A beautiful analytic fcn is the Riemann-Zeta fcn, which for $Re(s)>1$ is given by

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}$$

(if $s = x+iy$ then $n^s = n^x n^{iy} = n^x e^{iy \ln n}$)

note, for $x>1$, $\sum_{n \in \mathbb{N}} \frac{1}{|n^s|} = \sum_{n \in \mathbb{N}} \frac{1}{n^x}$ converges,

and for $x \geq 1+\delta$ this absolute convergence is uniform so also the partial sums

$$S_N(s) = \sum_{n=1}^{N} \frac{1}{n^s} \to \zeta(s) \text{ uniformly}$$

Thus by Morera,

$\zeta(s)$ is analytic for $Re\, s > 1$.

In fact, except for at $s=1$ ($\zeta(1) = \sum_{n \in \mathbb{N}} \frac{1}{n} = +\infty$),

$\zeta(s)$ extends to be analytic (which different formulas when $Re\, s < 1$) in the entire complex plane.

The Riemann zeta function has surprising connections to number theory, in particular the prime number theorem.

The Riemann hypothesis, that all the zeroes of $\zeta(s)$ lie on the line $Re\, s = 1/2$, is one of the great unsolved hypothese in Mathematics, see e.g. the Millenium prizes

this is a great topic area for a research report in our course.

more apps of C.I.F...

Mean Value Property Let $cl(D(z_0;R)) \subset A$
$f : A \to \mathbb{C}$ analytic.

then for $\gamma(\theta) = z_0 + Re^{i\theta}$, $0 \le \theta \le 2\pi$,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta$$

pf: $f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{z_0 - \zeta}\, d\zeta$. Compute!

Mean Value Property for harmonic functions

Corollary Let $u(x,y)$ be harmonic and C^2 in A, with $\{(x,y) \mid \|(x,y)-(x_0,y_0)\| \le R\} \subset A$.

then $$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(x_0 + R\cos\theta, y_0 + R\sin\theta\right)\, d\theta$$

pf: use harmonic conjugate theory.

these mean value properties have interesting consequences...