Math 4200

Wed 11/9

§ 4.1–4.2

§ 4.1: computing residues... these are the only ideas you need to construct the book's table

Let $f(z) = \frac{g(z)}{h(z)}$ with g, h analytic in $D(z_0;r)$ and $h(z_0) = 0$

1) easiest case (nice denominator i.e. polynomial) $\quad f(z) = \frac{f_1(z)}{(z-z_0)^k} + f_2(z)$ $\quad$ with f_1, f_2 analytic in $D(z_0;r)$ and $f_1(z_0) \neq 0$

$$\Rightarrow f_1(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$\operatorname{res}(f; z_0) = a_{k-1} \quad \text{(why?)}$$

$$\left(= \frac{1}{(k-1)!} f_1^{(k-1)}(z_0) \right)$$

example $\quad \operatorname{Res}\left(\frac{\sin 2z}{z^4} ; 0 \right) =$

2) general case $\quad f(z) = \frac{g(z)}{h(z)}$

$$g(z) = \sum_{n=M}^{\infty} a_n (z-z_0)^n \quad a_M \neq 0 \; ; \quad g(z) = (z-z_0)^M \sum_{n=M}^{\infty} a_n (z-z_0)^{n-M}$$

$$h(z) = \sum_{n=N}^{\infty} \tilde{a}_n (z-z_0)^n \quad a_N \neq 0 \; ; \quad h(z) = (z-z_0)^N \sum_{n=N}^{\infty} \tilde{a}_n (z-z_0)^{n-N}$$

$$\Rightarrow f(z) = \frac{(z-z_0)^M}{(z-z_0)^N} \, \frac{\sum_{n=M}^{\infty} a_n (z-z_0)^{n-M}}{\sum_{n=N}^{\infty} \tilde{a}_n (z-z_0)^{n-N}} \qquad a_M \neq 0, \quad \tilde{a}_N \neq 0$$

- $M \geq N \Rightarrow$ removable singularity; zero residue
- $N > M \Rightarrow \underbrace{(z-z_0)^{N-M}}_{(z-z_0)^k} f(z) = \frac{\cancel{(z-z_0)^N}}{\cancel{(z-z_0)^N}} \, \frac{\sum_{M}^{\infty} a_n (z-z_0)^{n-M}}{\sum_{n=N}^{\infty} \tilde{a}_n (z-z_0)^{n-N}} := \phi(z)$ is analytic @ z_0, with $\phi(z_0) \neq 0$

$$= \sum_{n=0}^{\infty} c_n (z-z_0)^n \qquad \operatorname{res}(f; z_0) = c_{k-1} = \frac{1}{(k-1)!} \phi^{(k-1)}(z_0)$$

the residue at z_0, $\operatorname{Res}(g/h; z_0)$ *is given by*

$$\operatorname{Res}(g/h; z_0) = \left[\frac{k!}{h^{(k)}(z_0)}\right]^k \times \begin{vmatrix} \frac{h^{(k)}(z_0)}{k!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(k+2)}(z_0)}{(k+2)!} & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{h^{(2k-1)}(z_0)}{(2k-1)!} & \frac{h^{(2k-2)}(z_0)}{(2k-2)!} & \frac{h^{(2k-3)}(z_0)}{(2k-3)!} & \cdots & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!} \end{vmatrix},$$

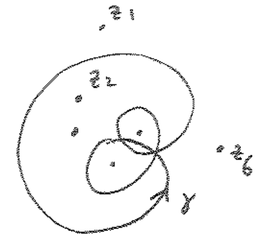
where the vertical bars denote the determinant of the enclosed $k \times k$ *matrix.*

Table 4.1.1 Techniques for Finding Residues

In this table g and h are analytic at z_0 and f has an isolated singularity. The most useful and common tests are indicated by an asterisk.

Function		Test	Type of Singularity	Residue at z_0
1.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)=0$	removable	0
*2.	$\frac{g(z)}{h(z)}$	g and h have zeros of same order	removable	0
*3.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)=0$ exists and is $\neq 0$	simple pole	$\lim_{z\to z_0}(z-z_0)f(z)$
*4.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0)=0,$ $h'(z_0)\neq 0$	simple pole	$\frac{g(z_0)}{h'(z_0)}$
5.	$\frac{g(z)}{h(z)}$	g has zero of order k, h has zero of order $k+1$	simple pole	$(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$
*6.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0$ $h(z_0)=0=h'(z_0)$ $h''(z_0)\neq 0$	second-order pole	$2\frac{g'(z_0)}{h''(z_0)}-\frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$
*7.	$\frac{g(z)}{(z-z_0)^2}$	$g(z_0)\neq 0$	second-order pole	$g'(z_0)$
*8.	$\frac{g(z)}{h(z)}$	$g(z_0)=0, g'(z_0)\neq 0,$ $h(z_0)=0=h'(z_0)$ $=h''(z_0), h'''(z_0)\neq 0$	second-order pole	$3\frac{g''(z_0)}{h'''(z_0)}-\frac{3}{2}\frac{g'(z_0)h^{(iv)}(z_0)}{[h'''(z_0)]^2}$
9.	$f(z)$	k is the smallest integer such that $\lim_{z\to z_0}\phi(z_0)$ exists where $\phi(z)=(z-z_0)^k f(z)$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$
*10.	$\frac{g(z)}{h(z)}$	g has zero of order l, h has zero of order $k+l$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$ where $\phi(z)=(z-z_0)^k\frac{g}{h}$
11.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0)=$ $\ldots=h^{k-1}(z_0)$ $=0, h^k(z_0)\neq 0$	pole of order k	see Proposition 4.1.7.

Different versions of the residue theorem



__Version 1__ (yesterday): $f : A \setminus \{z_1, .. z_N\} \to \mathbb{C}$ analytic.

γ contractible in A

$$\Rightarrow \int_\gamma f(\zeta)d\zeta = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(f; z_k)\, I(\gamma; z_k)$$

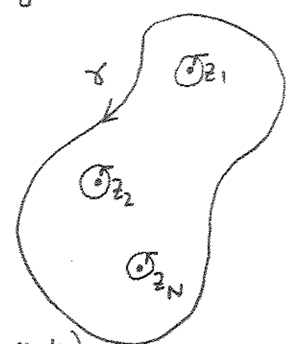
__Version 2__ special case via Green's Thm version of Cauchy's Thm:

$\gamma = \partial\Omega$, oriented c.c.

Ω bounded, $cl(\Omega) \subset A$

$z_1, .. z_N \in \Omega$

$f : A \setminus \{z_1 .. z_n\} \to \mathbb{C}$ analytic



$$\oint_\gamma f(\zeta)d\zeta = \sum_{k=1}^{n} \oint_{|\zeta - z_k| = \varepsilon_k} f(\zeta)d\zeta \quad \text{(Green's thm)}$$

$$= \sum_{k=1}^{n} \operatorname{Res}(f; z_k) \cdot 2\pi i$$

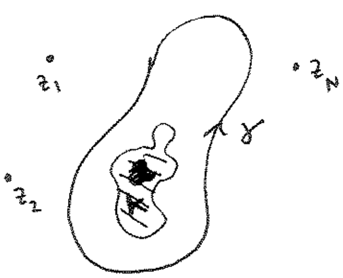
because $f(z) = \sum_{n=0}^{\infty} a_n^{(k)} (z - z_0)^k + \sum_{m=1}^{\infty} \frac{b_m^{(k)}}{(z-z_0)^m}$ converges uniformly on $|z - z_k| = \varepsilon_k$.

$$\int_\gamma f(\zeta)d\zeta = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k) \quad \blacksquare$$

__Version 3__ (exterior singularities)

$\gamma = \partial\Omega$, $K \subset \Omega$

$f : \mathbb{C} \setminus \{K \cup \{z_1, z_2 .. z_N\}\} \to \mathbb{C}$ analytic.

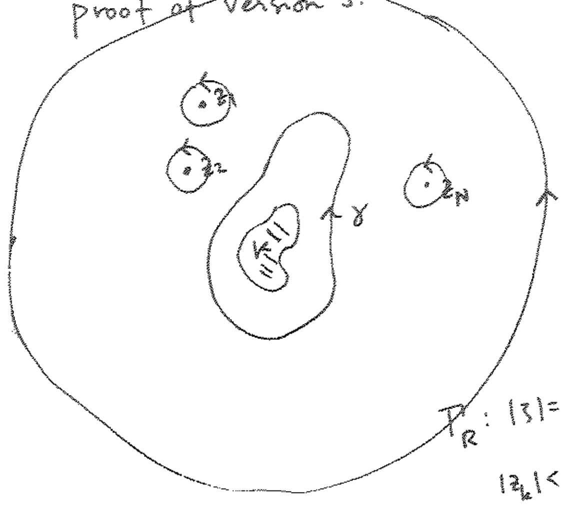


$$\oint_\gamma f(\zeta)d\zeta = -2\pi i \left(\sum_{k=1}^{n} \operatorname{Res}(f, z_k) + \operatorname{Res}(f; \infty) \right)$$

where $\operatorname{Res}(f; \infty) := \operatorname{Res}\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right)$.

proof on next page.

proof of version 3:



$\Gamma_R: |\zeta| = R.$

$|z_k| < R \ \forall k.$

Green's Thm:

$$\oint_{\Gamma_R} f(\zeta)d\zeta = \oint_{\gamma} f(\zeta)d\zeta + \underbrace{\sum_{k=1}^{n} \oint_{|\zeta - z_k| = \varepsilon_k} f(\zeta)d\zeta}_{2\pi i \sum_k \operatorname{Res}(f;z_k)}$$

Substitute

$$\zeta = \frac{1}{z}$$
$$d\zeta = -\frac{1}{z^2}dz$$

if $\zeta = Re^{it}$ $\quad 0 \le t \le 2\pi$
then
$$z = \frac{1}{\zeta} = \frac{1}{R}e^{-it}$$

So
$$\oint_{\Gamma_R} f(\zeta)d\zeta = \underbrace{\oint_{|z| = \frac{1}{R}} f\left(\tfrac{1}{z}\right)\left(-\tfrac{1}{z^2}dz\right)}_{\substack{\text{only sing @ } z=0, \\ \text{use Residue thm}}}$$

$$= 2\pi i \operatorname{Res}\left(-\tfrac{1}{z^2}f\left(\tfrac{1}{z}\right);0\right)\underbrace{(-1)}_{\text{index}}$$

$$= 2\pi i \operatorname{Res}\left(+\tfrac{1}{z^2}f\left(\tfrac{1}{z}\right);0\right)$$

$$= -2\pi i \operatorname{Res}(f;\infty) \quad \text{by def'n.}$$

Thus the Green's Thm Cauchy Thm result reads

$$\oint_{\gamma} f(\zeta)d\zeta = -2\pi i\left(\operatorname{Res}(f;\infty) + \sum_{k=1}^{N}\operatorname{Res}(f;z_k)\right)$$

■

Side comment: Complex change of variables in contour integrals:

$$\int_{\gamma} f(\zeta)d\zeta = \int_{g^{-1}(\gamma)} f(g(z))\,g'(z)dz$$

$\zeta = g(z)$ $\quad$ g, g^{-1} analytic

$z = g^{-1}(\zeta)$ $\quad$ g^{-1} analytic on γ

check: parameterize $g^{-1}(\gamma)$ by $\varphi(t)$, $a \le t \le b$

so γ is parameterized by $g(\varphi(t))$

$$\text{LHS} = \int_a^b f(g(\varphi(t)))\underbrace{g'(\varphi(t))\,\varphi'(t)}_{\frac{d}{dt}g(\varphi(t))}dt$$

(chain rule for curves)

$$\text{RHS} = \int_a^b f(g(\varphi(t)))\,g'(\varphi(t))\underbrace{\varphi'(t)dt}_{dz}$$

EQUAL!

Example Use the residue thm for exterior domains to compute

$$\oint_{|z|=2} \frac{3z^2+7}{z^3+2z-3}\, dz$$

first show that $z=\infty$ is the only singularity in the exterior domain

Check your answer by being clever, instead.