Math 4200
Mon Nov. 7

§ 4.1 & 4.2 Residues and the residue theorem : Back to contour integrals, but using Laurent series ideas.

Partial contour integral theorems list, so far:

<u>Cauchy Theorem</u> $f: A \to \mathbb{C}$ analytic, γ p.w C^1 closed contour homotopic to a point in A

$$\Rightarrow \int_\gamma f(\zeta)\,d\zeta = 0$$

(well, this is actually one of the deformation thms.)

[rectangle lemma
local antiderivatives
homotopy theorems
global antiderivatives in simply connected A.

<u>Index Theorem</u> γ p.w. C^1 closed contour, $z \in \mathbb{C}$

$$\Rightarrow I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\,d\zeta$$

↑ topological winding #
↑ how to compute index with a contour integral

<u>Cauchy integral formula</u> $f: A \to \mathbb{C}$ analytic, γ p.w. C^1 closed contour in A, $z \notin \gamma$
γ homotopic to a point in A

$$\Rightarrow f(z)\, I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

[proof: apply Cauchy Theorem to $g(\zeta) := \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$

[also get differentiation theorems.
also get Taylor & Laurent series

NEW TODAY:

§ 4.2 <u>Residue Theorem</u>

$f: A \to \mathbb{C}$ analytic except at (isolated) points $\{z_1, z_2, \ldots z_N\} \subset A$

γ p.w. C^1 curve in γ contractible to a point in A

$Res(f; z_j)$, the residue of f at z_j, i.e. the coefficient of $\frac{1}{(z - z_j)}$, in the Laurent series for f at z_j (the "b_1")

$$\int_\gamma f(\zeta)\,d\zeta = 2\pi i \sum_{j=1}^{N} Res(f; z_j)\, I(\gamma; z_j)$$

[notice this Theorem contains the Cauchy theorem and Cauchy Integral formula as special cases ... of course the proof of the Residue Theorem also follows from these results

<u>Exercise 1</u> A lot of our old examples for computing contour integrals illustrate the Residue Theorem (which will also be useful for lots of new examples).

- Use the Residue Theorem and partial fractions to compute

$$\oint_{|z|=2} \frac{1}{z^2-1}\,dz$$

- How about

$$\oint_{|z|=2} \frac{z}{z^2-1}\,dz$$

(What if we wanted $\displaystyle\oint_{|z|=2} \frac{e^z}{z^2-1}\,dz$, or $\displaystyle\oint_{|z|=2} \frac{z^{10}}{z^2-1}\,dz$?)

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proof of residue theorem:
<u>at each</u> z_k, f has Laurent expansion

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n^{(k)}(z-z_k)^n}_{\substack{\text{some pos.}\\ \text{rad of conv.}}} + \underbrace{\sum_{m=1}^{\infty} b_m^{(k)}\left(\frac{1}{z-z_k}\right)^m}_{\substack{\text{conv for } |z-z_k|>0,\\ \text{unif, abs for}\\ |z-z_k|\geq\varepsilon>0}}$$

$$f(z) = S_1^{(k)}(z) + S_2^{(k)}(z)$$
near z_k

Thus

$$f(z) - \sum_{k=1}^{N} S_2^{(k)}(z)$$

has removable singularities at each z_k, so extends to be analytic in A

Cauchy Theorem $\Rightarrow$ $\displaystyle\int_\gamma \left(f(z) - \sum_{k=1}^{N} S_2^{(k)}(z)\right) dz = 0$

$$\Rightarrow \int_\gamma f(z)\,dz = \sum_{k=1}^{N} \int_\gamma S_2^{(k)}(z)\,dz = \sum_{k=1}^{N}\left(\int_\gamma \sum_{m=1}^{\infty} b_m^{(k)} \frac{1}{(z-z_k)^m}\,dz\right)$$

$$= \sum_{k=1}^{N} 2\pi i\, b_1^{(k)}\, I(\gamma; z_k) \quad \blacksquare$$

[by $\int, \Sigma$ interchange & FTC when $m \neq 1$]

So, in order to compute contour integrals around γ it becomes important to find efficient ways to compute residues at isolated singularities. This is the goal of section 4.1 ... you probably don't want to compute the entire Laurent series just to get the residue.

What will often work, in case $f(z) = \dfrac{g(z)}{h(z)}$, with g, h analytic near z_0:

is to use Taylor series ideas:

Case I $\quad g(z_0) \neq 0$
$\quad h(z_0) = 0, \; h'(z_0) \neq 0 \qquad$ (zero of order 1)

$$g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad a_0 \neq 0$$

$$h(z) = \sum_{n=1}^{\infty} b_n (z-z_0)^n \qquad b_1 \neq 0$$

$$\Rightarrow (z-z_0)\frac{g(z)}{h(z)} = \frac{(z-z_0)a_0 + (z-z_0)^2 a_1 + \cdots}{(z-z_0)b_1 + (z-z_0)^2 b_2 + \cdots}$$

$$= \frac{\cancel{z-z_0}}{\cancel{z-z_0}}\left[\frac{a_0 + a_1(z-z_0) + \cdots}{b_1 + b_2(z-z_0) + \cdots}\right] \qquad \begin{array}{l} a_0 \neq 0 \\ b_1 \neq 0 \end{array}$$

$\Rightarrow (z-z_0)\dfrac{g(z)}{h(z)}$ extends as analytic @ z_0

$$= c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots \qquad c_0 = \frac{a_0}{b_1}$$

$$\Rightarrow \frac{g(z)}{h(z)} = \frac{a_0/b_1}{z-z_0} + c_1 + c_2(z-z_0)^2 + \cdots$$

$$\boxed{\text{simple pole; residue } = a_0/b_1 = \frac{g(z_0)}{h'(z_0)}} \qquad \begin{array}{l} g(z_0) \neq 0 \\ h(z_0) = 0 \\ h'(z_0) \neq 0. \end{array}$$

Exercise 2 $\quad$ Do all four contour integrals on page 2 Exercise 1, using this shortcut

e.g., the first one goes like this: $\displaystyle\oint_{|z|=2} \frac{1}{z^2-1}\,dz = 2\pi i\left(\mathrm{Res}(f;1) + \mathrm{Res}(f;-1)\right)$

$$= 2\pi i\left(\frac{1}{2} + \frac{1}{-2}\right) = 0$$

$$f(z) = \frac{g(z)}{h(z)} = \frac{1}{z^2-1} \qquad h'(z) = 2z$$

There are other formulas for more complicated poles, see page 250 of text.
For essential singularities (like in Friday example), things could get difficult.

Exercise 3
Can you derive the formula for $\mathrm{Res}(f; z_0)$ in case $f(z) = \frac{g(z)}{h(z)}$ $\quad g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, $h''(z_0) \neq 0$?

(pole of order 2).

The answer is $\mathrm{Res}(f; z_0) = 2\,\dfrac{g'(z_0)}{h''(z_0)} - \dfrac{2}{3}\,\dfrac{g(z_0)\,h'''(z_0)}{[h''(z_0)]^2}$:)

hint: consider
$$(z-z_0)^2 \frac{g(z)}{h(z)} = \frac{\cancel{(z-z_0)^2}\,\left(a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right)}{\cancel{(z-z_0)^2}\,\left(b_2 + b_3(z-z_0) + \cdots\right)} \qquad a_0 \neq 0,\ b_2 \neq 0$$

$$= \frac{S_1(z)}{S_2(z)}$$

$$(z-z_0)^2 \frac{g(z)}{h(z)} = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$