Math 4200-1
Fri 11/4

§3.3 characterizing isolated zeroes. multiplying Laurent series

HW for Fri 11/11
3.3 9, 17, 20b.
4.1 1de, 3, 5, 7, 9
4.2 2, 3, 4, 6, 8, 13, 15
(Exam 2 is Wed 11/16, covers thru §4.2)



Isolated singularities table:
$f$ is analytic in $D(z_0;r)\setminus\{z_0\}$, some $r>0$

| Type of isolated singularity | Laurent expansion definition | characterization in terms of $\lim_{z\to z_0} f(z)$ |
|---|---|---|
| removable (because $f$ extends to be analytic in $D(z_0;r)$) | $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ (no negative powers in Laurent) | ① $\lim_{z\to z_0} f(z)$ $\exists$, as a finite #, or ② $\lvert f(z)\rvert \le M$ $\forall$ $\lvert z-z_0\rvert\le\rho$, some $0<\rho<r$, or ③ $\lim_{z\to z_0} f(z)(z-z_0)=0$ |
| pole (North pole!) of order N; simple pole if N=1 | $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{N} \frac{b_m}{(z-z_0)^m}$ with $b_N\neq 0$ | ① $\lim_{z\to z_0} f(z)=\infty$, (the North pole!) or ② $\exists$ N s.t. $g(z):=(z-z_0)^N f(z)$ has a removable singularity @ $z=z_0$, with $g(z_0)\neq 0$ |
| essential singularity | $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{m=1}^{\infty}\frac{b_m}{(z-z_0)^m}$ s.t. $\exists$ $m_j\to\infty$ s.t. $b_{m_j}\neq 0$ | $\forall\ 0<\rho<r$, $\overline{f(D(z_0,\rho)\setminus\{z_0\})}=\mathbb{C}$ [In fact, more is true, called Picard's Theorem: $f(D(z_0,\rho)\setminus\{z_0\})$ contains all of $\mathbb{C}$, except for at most one point – e.g. $f(z)=e^{1/z}$, $z_0=0$]. |

Today, we explain column 3 characterizations.

$\boxed{\text{removable singularity}}$ : If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ (Laurent characterization)

in $D(z_0, r) \setminus \{z_0\}$

then since this power series also converges at $z_0$, it defines an analytic function in $D(z_0 r)$

$\Rightarrow$ 1 $\lim_{z \to z_0} f(z) = f(z_0) = a_0$ exists (since analytic $\Rightarrow$ continuous)

$\Rightarrow$ 2 $f$ bounded near $z_0$; in fact for $M = |a_0| + 1$ $\exists\, \delta > 0$ s.t. $0 < |z - z_0| < \delta \Rightarrow |f(z)| \le M$

$\Rightarrow$ 3 $\lim_{z \to z_0} |f(z)(z-z_0)| \le \lim_{z \to z_0} M |z - z_0| = 0$

so $\lim_{z \to z_0} f(z)(z-z_0) = 0$

The circle is completed if we show 3 $\Rightarrow$ Laurent characterization

recall, if $\gamma(t) = \rho e^{it}$ $\quad 0 < \rho < r$, $\quad 0 \le t \le 2\pi$

then $m \geq \cancel{1}$, $b_m = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\zeta - z_0)^{m-1} \, d\zeta$ (Wednesday notes)

3 $\Rightarrow b_m = 0 \;\forall m$:

$$|b_m| \le \frac{1}{2\pi} \int_\gamma |f(\zeta)| \, \rho^{m-1} |d\zeta|$$

$$\le \underbrace{\frac{1}{2\pi} \rho^{m-1} 2\pi}_{\substack{\to 0 \ (m>1) \\ \text{cont } (m=1)}} \underbrace{\rho \max \{ |f(\zeta)|, |\zeta - z_0| = \rho \}}_{\substack{\to 0 \text{ as } \rho \to 0 \\ \text{by } 3}}$$

∎

$\boxed{\text{pole}}$

Laurent definition

$\Rightarrow g(z) := (z-z_0)^N f(z)$

has only ~~•~~ non-negative terms in it's Laurent expansion, so extends to be analytic at $z_0$, with

2 $\quad g(z_0) = b_N \ne 0$

2 $\Rightarrow$ 1 since $\lim_{z \to z_0} |f(z)|$

$= \lim_{z \to z_0} \underset{\substack{\downarrow \\ \infty.}}{\frac{1}{|z-z_0|^N}} \underset{\substack{\downarrow \\ |g(z_0)| \ne 0}}{|g(z)|}$

So it remains to show

1 $\Rightarrow$ Laurent def of pole.

1 $\Rightarrow$ Laurent def of pole:

$$\lim_{z \to z_0} f(z) = \infty.$$

let $k(z) = \frac{1}{f(z)}$

$\Rightarrow \lim_{z \to z_0} k(z) = 0$ $\Rightarrow$ $k(z)$ has a removable singularity at $z_0$

$\Rightarrow k(z) = \sum_{n=N}^{\infty} c_n (z-z_0)^n$ $\quad c_N \neq 0,\ N \geq 1$ (since $k(z_0)=0$) in some $D(z_0,\rho)$

$k(z) = (z-z_0)^N h(z) \qquad h(z_0) \neq 0$

$\Rightarrow f(z) = \frac{1}{k(z)} = \frac{1}{(z-z_0)^N} \underbrace{\frac{1}{h(z)}}$

analytic near $z_0$, so has Taylor series $\sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad a_0 \neq 0$

i.e. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n-N} \qquad a_0 \neq 0$ ∎

essential singularity:

By logic (!), it suffices to show that if it is not true that

$$\forall\, 0<\rho<r, \quad \overline{f(D(z_0,r)\setminus\{z_0\})} = \mathbb{C}$$

then $z_0$ is either a pole or a removable singularity !!

So, assume

$$\exists\, 0<\rho<r \quad \text{with} \quad \overline{f(D(z_0,\rho)\setminus\{z_0\})} \neq \mathbb{C}$$

i.e. $\exists\ w_0, \varepsilon$ s.t. $D(w_0;\varepsilon) \cap \overline{f(D(z_0;\rho)\setminus\{z_0\})} = \emptyset$.

Define $k(z) := \frac{1}{f(z)-w_0}$

$|k(z)| \leq \frac{1}{\varepsilon} \qquad \forall\, z \in D(z_0,\rho)\setminus\{z_0\}$

$\Rightarrow k$ has removable sing @ $z_0$

$k(z) = \sum_{n=N}^{\infty} a_n (z-z_0)^n \qquad N \geq 0,\ a_N \neq 0$

$\frac{1}{f(z)-w_0} = \sum_{n=N}^{\infty} a_n (z-z_0)^n \qquad a_N \neq 0$

$= (z-z_0)^N h(z) \qquad h(z_0) \neq 0$

$f(z) - w_0 = \frac{1}{(z-z_0)^N} \tilde{h}(z)$, $\quad \tilde{h}(z) = \frac{1}{h(z)}$ analytic @ $z_0$.

$f(z) = w_0 + \frac{1}{(z-z_0)^N} \tilde{h}(z)$ .... i.e. $f$ has a removable singularity or a pole at $z_0$ ∎

Justification for computing Laurent series coefficients of a product, by term by term multiplication: (you actually used this in 3.3.6)

<u>Theorem</u> Let $f(z)$, $g(z)$ have Laurent series in $A = \{z \mid r_1 < |z-z_0| < r_2\}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{c_m}{(z-z_0)^m} := \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$g(z) = \sum_{k=-\infty}^{\infty} b_k (z-z_0)^k$$

Then $f(z)g(z)$ has Laurent Series

$$f(z)g(z) = \sum_{n=-\infty}^{\infty} d_n (z-z_0)^n$$

where
$$d_n = \lim_{M,N\to\infty} \sum_{j=-M}^{N} a_j b_{n-j} := \sum_{j=-\infty}^{\infty} a_j b_{n-j}$$

<u>Proof</u>: Recall we recover $d_n$ with a contour integral; for $r_1 < r < r_2$

Fix $n \in \mathbb{Z}$.

$$* \qquad d_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(\zeta)g(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$$

(or any other index 1 curve about $z_0$)

Let
$$f_{M,N}(z) = \sum_{j=-M}^{N} a_j (z-z_0)^j, \qquad g_{M,N}(z) = \sum_{k=-N+n}^{M+n} b_k (z-z_0)^k$$

$$\left.\begin{matrix} f_{M,N} \to f \\ g_{M,N} \to g \end{matrix}\right\} \text{uniformly, as } M,N \to \infty, \text{ on } \gamma = \{z \mid |z-z_0| = r\}$$

$\Rightarrow f_{M,N}\, g_{M,N} \to fg$ uniformly on $\gamma$

$\Rightarrow$ (via $*$), that $d_n$ is the limit as $M,N \to \infty$, of the $n^{th}$ Laurent coeff of $f_{M,N}\, g_{M,N}$. But this product is a product of <u>finite</u> sums, and the (unique) coeff of $z^n$ in its Laurent series is precisely the finite sum obtained by term by term multiplication

$$\sum_{j=-M}^{N} a_j b_{n-j}$$

$\blacksquare$

example: Your hw problem 3.3.6:

$$f(z) = e^{\frac{1}{z}}\left(\frac{1}{1-z}\right) \quad @ \; z_0 = 0$$

So, what is $\oint_{|z|=1} f(z)\,dz$ ?