Math 4200
Wed 11/30

§5.1–5.2 We started talking about fractional linear transformations:

FLT's

$f(z) = \dfrac{az+b}{cz+d}$ with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ [if convenient can normalize so that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$, by dividing original numerator & denom by $\sqrt{\det}$]

$f: S^2_R \to S^2_R$

$S^2_R = \mathbb{C} \cup \infty$

amazing "$SL(2, \mathbb{C})$ group action property"

if $g(w) = \dfrac{\alpha w + \beta}{\gamma w + \delta}$, then $g \circ f(z) = \dfrac{Az+B}{Cz+D}$ with $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

"matrix" of composition is product of the matrices".

- thus $f^{-1}(z) = \dfrac{dz - b}{-cz + a}$
- $f'(z) = \dfrac{ad-bc}{(cz+d)^2} \neq 0$ $\quad (z \neq -\frac{c}{d})$ so FLT's are conformal.

<u>Exercise 7</u> FLT's map circles & lines to circles or lines

pf: True for
$T_1(z) = z + a$ (translations)
$T_2(z) = cz$ (scaling and rotation)
$T_3(z) = \frac{1}{z}$ (inversion)

$\left.\begin{matrix} T_1, T_2 \end{matrix}\right\}$ circles → circles, lines → lines

$\left.\begin{matrix} T_3 \end{matrix}\right\}$ circles thru 0 → lines, lines not thru 0 → circles

7a) Use the fact that lines and circles are implicitly defined by equations of the form
$$A(x^2+y^2) + Bx + Cy + D = 0$$
to check claim for T_3

7b) Show every FLT is a composition of type T_1, T_2, T_3's FLT's.

- Notice that for $a, b, c \in \mathbb{C}$ distinct

$$f(z) = \frac{z-a}{z-b}\left(\frac{c-b}{c-a}\right)$$

maps
$$a \mapsto 0$$
$$b \mapsto \infty$$
$$c \mapsto 1$$

In fact f is the unique FLT which maps $\begin{matrix} a \to 0 \\ b \to \infty \\ c \to 1 \end{matrix}$ since any such FLT must equal $\tilde{C}\left(\frac{z-a}{z-b}\right)$ with $\tilde{C}\left(\frac{c-a}{c-b}\right) = 1$.

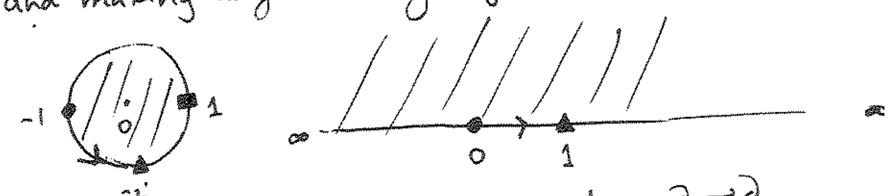
- Thus, if we want an FLT to map $\begin{matrix} a \to \alpha \\ b \to \beta \\ c \to \gamma \end{matrix}$ let $g(w) = \frac{w-\alpha}{w-\beta} \; \frac{\gamma-\beta}{\gamma-\alpha}$

Then $g^{-1} \circ f(z) : \begin{matrix} a \to 0 \to \alpha \\ b \to \infty \to \beta \\ c \to 1 \to \gamma \end{matrix}$ does the trick (and is the unique FLT with this property)

$\sim$ this FLT will have matrix $[g]^{-1}[f]$. Alternately you can solve $g(w) = f(z)$ for w. (!)

- Circles are uniquely determined by 3 points on them. Lines are uniquely determined by two finite points and ∞. Use this fact and the fact that $\{\text{circles, lines}\} \to \{\text{circles, lines}\}$ when mapping with FLT's to do

<u>Exercise 8</u> Find an FLT which maps the unit disk to the upper half plane, by mapping
$$-1 \to 0$$
$$1 \to \infty$$
$$-i \to 1$$
and making any necessary adjustments.



by magic thm once you know $\partial \to \partial$ you only need to check that one interior point goes to the correct side of the image arc... you can also do this by checking image orientation.

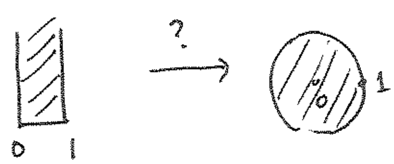
Exercise 9 Find the inverse transformation to the one in exercise 8

Exercise 10 Find all conformal diffeomorphisms of $D(0,1) \to D(0,1)$

Hint: Let $z_0 \in D(0,1)$.

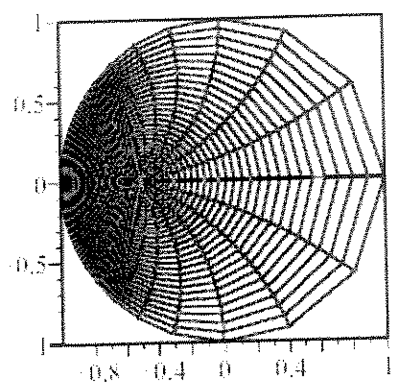
Consider $f(z) = e^{i\theta} \dfrac{z - z_0}{1 - \bar{z}_0 z}$ then appeal to max. modulus principle and uniqueness thm half of Riemann mapping thm.

Exercise 11 Find a conformal bijection of the half strip $\{z = x+iy \mid 0<x<1, y>0\}$ to the unit disk, using a composition of maps!



fractional linear transformations!

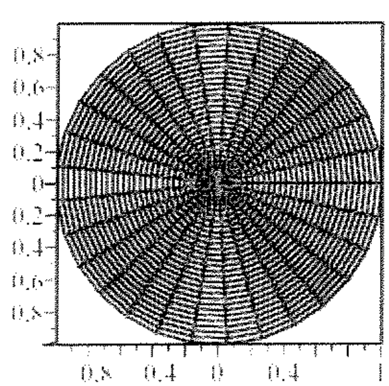
```
> with(plots):
> P := (r, θ) → r·exp(I·θ):
> Mob := z → (z - .5)/(1 - .5·z):
> plot3d([Re(Mob(P(r, θ))), Im(Mob(P(r, θ))), 0], r = 0..1, θ = 0..evalf(2·Pi), grid
      = [30, 30], orientation = [-90, 0], style = wireframe, color = black, axes = boxed);
```



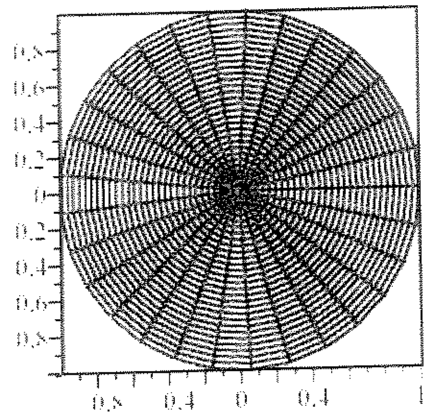
$f(z) = \dfrac{z + \frac{1}{2}}{1 + \frac{1}{2}z}$

$g(z) = \dfrac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$

```
> plot3d([Re(P(r, θ)), Im(P(r, θ)), 0], r = 0..1, θ = 0..evalf(2·Pi), grid = [30, 30],
      orientation = [-90, 0], style = wireframe, color = black, axes = boxed);
```



```
> plot3d([ℜ(P(r, θ)), ℑ(P(r, θ)), 0], r = 0..1, θ = 0..evalf(2·π), grid = [30, 30], orientation
      = [-90, 0], style = wireframe, color = black, axes = boxed);
```



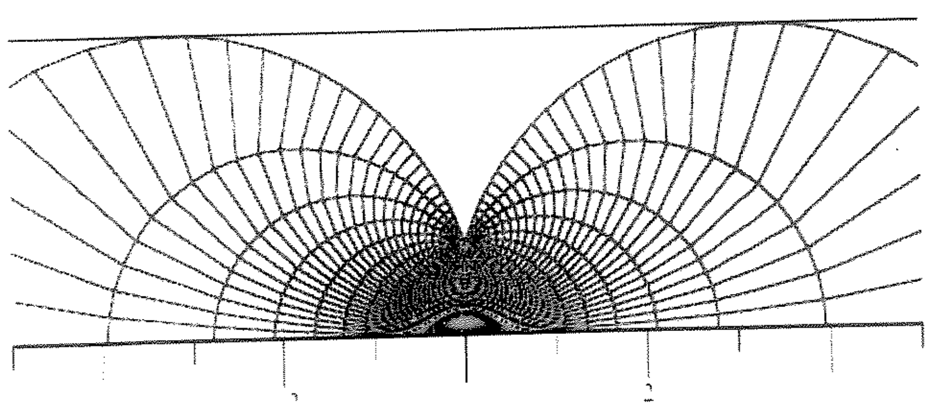
$f(z) = -i\left(\frac{z+1}{z-1}\right)$

$g(z) = \frac{-z+i}{-z-i}$

$= \frac{z-i}{z+i}$

```
>
> Mob2 := z → -I · (z + 1)/(z - 1) :

> plot3d([ℜ(Mob2(P(r, θ))), ℑ(Mob2(P(r, θ))), 0], r = 0..1, θ = (.3)..evalf((2·π
      - 0.3)), axes = boxed, grid = [30, 30], orientation = [-90, 0], style = wireframe, scaling
      = constrained, color = black);
```



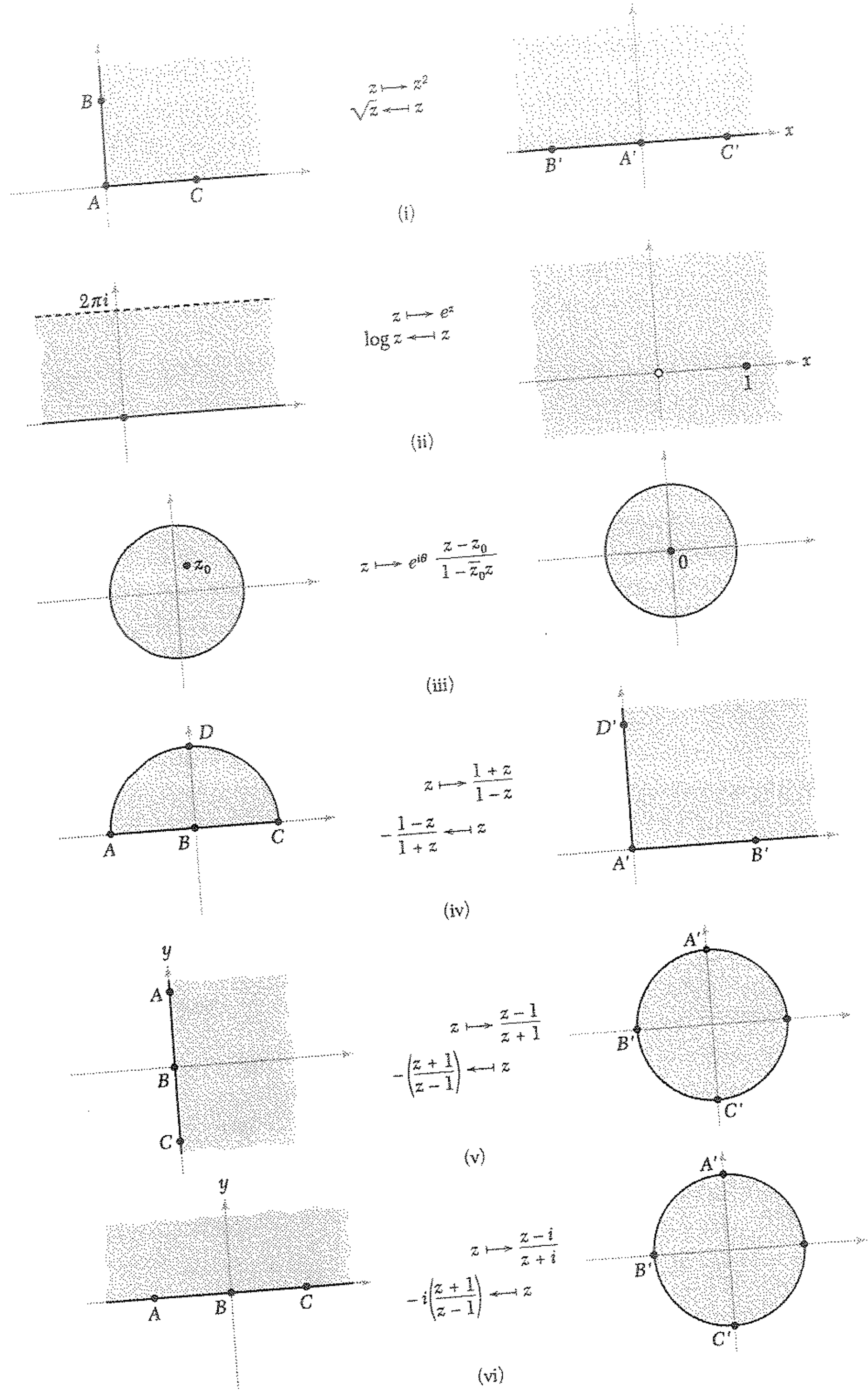


Figure 5.2.10: Some common transformations.

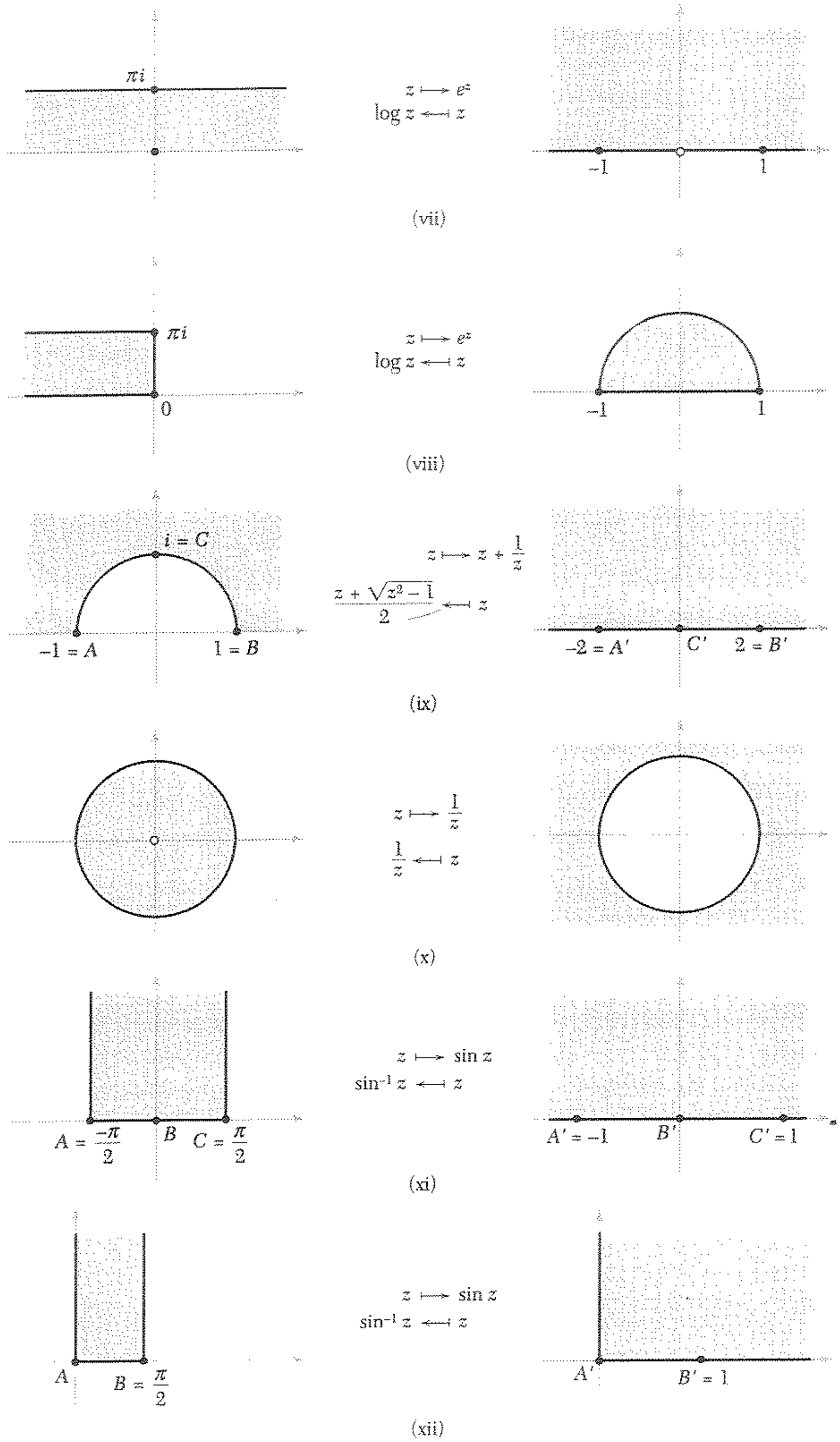


Figure 5.2.11: More common transformations.