Math 4200
Mon 11/28

Chapter 5 : Conformal maps

This is actually an in-depth return to ideas we began the course with.

Recall the chain rule for curves

f analytic @ z_0

$\gamma: I \to \mathbb{C}$ diff ble, $\gamma(t_0) = z_0$

$$\Rightarrow \frac{d}{dt} f\circ\gamma \Big|_{t=t_0} = f'(\gamma(t_0))\,\gamma'(t_0) = f'(z_0)\,\gamma'(t_0)$$

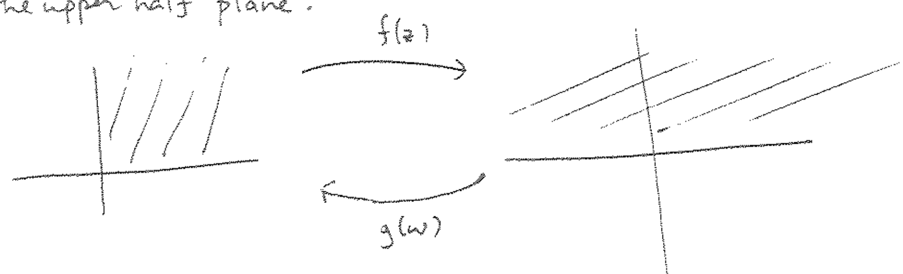
So the differential map $df_{z_0}: T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}$ rotates all tangent vectors by $\arg(f'(z_0))$ and scales them by $|f'(z_0)|$

We call (ed) a differentiable map conformal iff it has this infintesimal (i.e. on tangent vectors) rotation and scaling property $\forall z \in A$, the domain of f. By Cauchy Riemann this equivalent to $f: A \to \mathbb{C}$ analytic and $f'(z) \neq 0 \ \forall z \in A$

In Chapter 5 we are interested in finding bijective conformal maps between open connected domains A, B of $\mathbb{C}$. In such cases we call A and B conformally equivalent

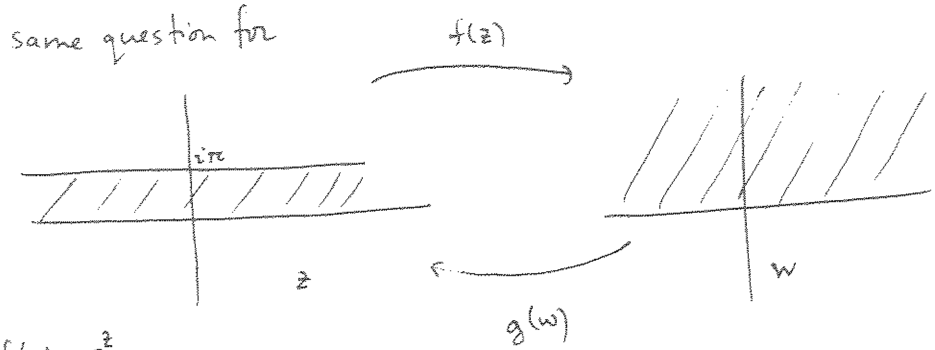
applications include PDE's and geometry.

Exercise 1 How many conformal bijections can you find between the 1st quadrant and the upper half plane?



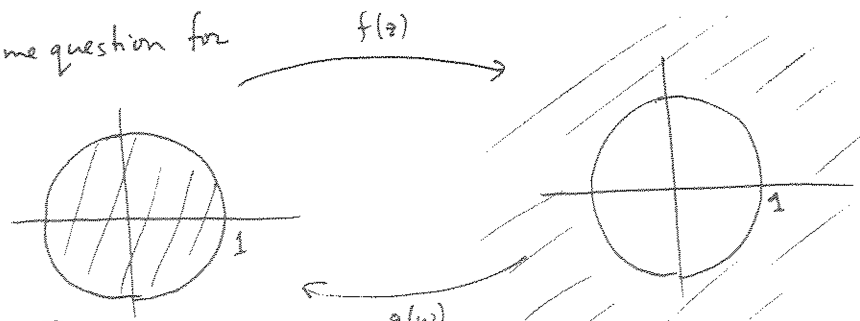
$f(z) = z^2$
$g(w) = \sqrt{w}$ is one pair

<u>Exercise 2</u> same question for



one pair: $f(z) = e^z$
$g(w) = \log w$

<u>Exercise 3</u> same question for



$f(z) = \frac{1}{z}$
$h(w) = \frac{1}{w}$ is one pair

($f(0) = \infty$, so this should really be considered as a map on the Riemann sphere)

Chances are we missed some examples...

Riemann Mapping Theorem (version 1)

Let $A \subset \mathbb{C}$ (but $A \neq \mathbb{C}$) be open and simply connected.
Let $z_0 \in A$.
Then $\exists!\ f: A \to D(0;1)$ s.t. f is a conformal bijection satisfying

$$\left.\begin{array}{l} f(z_0)=0 \\ f'(z_0) \text{ is real and positive} \end{array}\right\} \text{3 real degrees of freedom: } z_0,\ \arg f'(z_0).$$

Proof of existence is advanced topic (would've been a good project ☺)
But we have the tools to prove uniqueness:

Suppose f_1, f_2 satisfy the conditions above.
Consider $g(z) := f_2 \circ f_1^{-1} : D \to D$

$g(0)=0$

$g'(0) = f_2'(z_0)\, f_1^{-1\prime}(0) = f_2(z_0) \dfrac{1}{f_1'(z_0)} \in \mathbb{R}^+$

Since $g(0)=0$, $G(z) := \begin{cases} \frac{g(z)}{z} & z \neq 0 \\ g'(0) & z=0 \end{cases}$ has removable sing @ $z=0$ (is cont. there), so is analytic $G: D \to D$.

for $0<r<1$ on $|z|=r$ we have $|G(z)| \leq \frac{1}{r}$

Max princ $\Rightarrow |G(z)| \leq \frac{1}{r}\quad \forall\ |z| \leq r$

$r \to 1 \Rightarrow |G(z)| \leq 1 \quad \forall\ |z| < 1$

$\Rightarrow |g(z)| \leq |z| \quad \forall\ |z|<1$

but by symmetric reasoning,
for $g^{-1}(z) = f_1 \circ f_2^{-1}$, we have $|g^{-1}(z)| \leq |z| \quad \forall\ |z|<1$

$\Rightarrow \underbrace{|g^{-1}(g(z))|}_{= |z|} \leq |g(z)| \leq |z|$

$\Rightarrow |g(z)| = |z| \quad \forall\ z \in D$

$\Rightarrow |G(z)| \equiv 1 \Rightarrow G(z) = \text{constant}$ (also Maximum modulus principle)

$\Rightarrow \frac{g(z)}{z} = e^{i\theta}$

$\Rightarrow g(z) = e^{i\theta} z$

$g'(0) \in \mathbb{R}^+ \Rightarrow g(z) = z$

$\Rightarrow f_2(f_1^{-1}(z)) = z$

$\Rightarrow f_1^{-1}(z) = f_2^{-1}(z)$

$\Rightarrow f_1 = f_2$ ∎

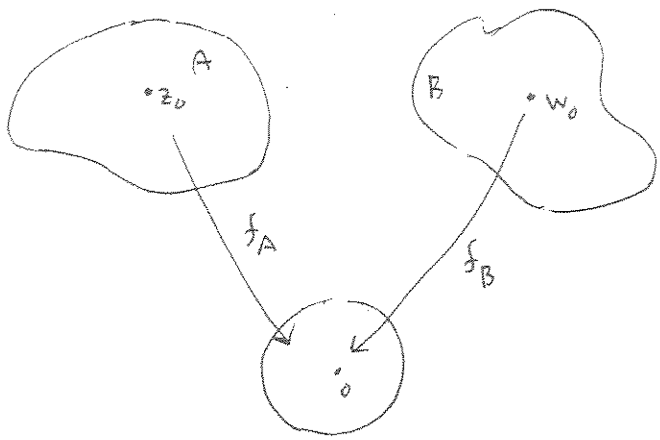
Riemann Mapping Theorem (version 2)

Let $A, B \subset \mathbb{C}$ be open & simply connected and not all of $\mathbb{C}$
Let $z_0 \in A$, $w_0 \in B$

Then $\exists!$ $f : A \to B$ conformal bijection s.t.

$$f(z_0) = w_0$$
$$f'(z_0) \in \mathbb{R}^+$$

pf: Chase diagram. Let f_A, f_B be as in version 1:



$\exists$: consider $f := f_B^{-1} f_A$

Check!

!: If g also satisfies conditions, compare say f_A to $f_B \circ g$

The maps we were missing in the earlier examples were fractional linear transformations (FLT's). We saw some of these already when we discussed the Möbius transformations of the unit disk....

an FLT $f(z)$, $f: \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ is defined by

$$f(z) = \frac{az+b}{cz+d} \qquad \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc \neq 0, \qquad a,b,c,d \in \mathbb{C}$$

($\det = 0$ $\Rightarrow f$ const!)

<u>Example</u> $f(z) = az+b = \dfrac{az+b}{0z+1}$. You will show in HW these are the only $\overset{\text{injective}}{\text{1-1}}$ conformal maps defined on $\mathbb{C}$!

<u>Exercise 4</u>: Why is there no conformal bijection $f: \mathbb{C} \to D(0;1)$?

Properties of FLT's:

<u>Exercise 5</u> Show that if

$$f(z) = \frac{az+b}{cz+d}$$

$$g(w) = \frac{\alpha w+\beta}{\gamma w+\delta}$$

Then $$g(f(z)) = \frac{Az+B}{Cz+D},$$

where $$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The matrix of the composition is the product of the matrices!

Geometers would say: The <u>group</u> $SL(2,\mathbb{C})$ ("$\underset{\det=1}{\underline{\text{Special}}}$ linear group of 2×2 matrices with $\mathbb{C}$ coeffs") <u>acts</u> on $\mathbb{C}\cup\{\infty\}$.

Exercise 6 Use exercise 5 to show that

if $f(z) = \dfrac{az+b}{cz+d}$ $\qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det A \neq 0$

then $f^{-1}(w) = \dfrac{dw-b}{-cw+a}$ $\qquad \mathrm{Adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Cor FLT's are bijections of $\mathbb{C} \cup \{\infty\}$. (you can check ∞ separately)

Notice if $f(z) = \dfrac{az+b}{cz+d}$

$$\text{then } f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0 \qquad (z \neq -\tfrac{c}{d})$$

so FLT's are conformal

Exercise 7 FLT's map circles & lines into circles or lines

pf: True for $T_1(z) = z + a$ (translation)

$T_2(z) = cz$ (scaling & rotation)

$T_3(z) = \frac{1}{z}$ (inversion)

7a) Use the fact that lines & circles are solution sets to equations of the form

$$A(x^2+y^2) + Bx + Cy + D = 0$$

to verify that T_3 makes $\{\text{circles, lines}\} \to \{\text{circles, lines}\}$ $\left(z = x+iy, \ \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)$

7b) Show every FLT is a composition of T_1, T_2, T_3 type FLT's.

Hint: if $f(z) = \dfrac{az+b}{cz+d}$ and $a, c \neq 0$, first do long division.

to be continued...

Notice that

$$f(z) = \left(\frac{z-a}{z-b}\right)\left(\frac{c-b}{c-a}\right)$$

maps $a \longmapsto 0$
$b \longmapsto \infty$
$c \longmapsto 1$

using this f and its inverse we can construct FLT's to ~~a~~ map

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longrightarrow \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

Since 3 points uniquely define circles one can use FLT's to map any circle or line to any other circle or line

Exercise 8. Find an FLT which maps the unit disk to the upper half plane, by mapping

$-1 \to 0$
$1 \to \infty$
$-i \to 1$

and making any necessary adjustments