Math 4200
Wed. 11/23

Laplace transform §8.1-8.2 (highlights)

HW from §5.1, 5.2:
5.1 7, 10, 11, 12
5.2 1, 4a, 6, 7, 10, 11, 24, 26, 33, 34
final HW assignment!

Recall (?), for $f:[0,\infty)\to\mathbb{C}$ (and we usually write $f(t)$), the Laplace transform

$$* \quad \mathcal{L}\{f(t)\}(z) := \int_0^\infty e^{-zt} f(t)\,dt$$

We restrict to f of <u>exponential order</u>, i.e. $\exists\, A, B \in \mathbb{R}$ s.t.

$$|f(t)| \le A e^{Bt} \quad \forall t \ge 0$$

Then, for $\mathrm{Re}\, z > B$, $*$ converges absolutely, since

$$\left|\int_0^N e^{-zt} f(t)\,dt\right| \le \int_0^N |e^{-zt} f(t)|\,dt \le \int_0^N e^{-t\,\mathrm{Re}\, z} A e^{Bt}\,dt$$
$$= \int_0^N e^{t(B-\mathrm{Re}\, z)} A\,dt < A\int_0^\infty e^{t(B-\mathrm{Re}\, z)}\,dt$$
$$= \frac{A}{B-\mathrm{Re}\, z}$$

Convention is to use capital letters for Laplace transform, i.e. to call $\mathcal{L}\{f(t)\}(z) = F(z)$.

examples

- $f(t) = e^{at}$

$$F(z) = \int_0^\infty e^{-zt} e^{at}\,dt = \int_0^\infty e^{(-z+a)t}\,dt = \left.\frac{e^{(a-z)t}}{a-z}\right]_0^\infty = 0 - \frac{1}{a-z} = \frac{1}{z-a}$$

$\mathrm{Re}\, z > a$
(for $\mathrm{Re}\, z \le a$ integral diverges)

- $f(t) = e^{ikt} = \cos kt + i \sin kt$

$$F(z) = \frac{1}{z-ik} \quad \text{(see above, for } \mathrm{Re}\, z > 0)$$
$$= \frac{1}{z-ik}\,\frac{z+ik}{z+ik} = \frac{z}{z^2+k^2} + i\,\frac{k}{z^2+k^2}$$

$F(z)$ is analytic in z (in general), and for $z = s$ real we have

$$\mathcal{L}\{\cos kt + i \sin kt\}(s)$$
$$\parallel$$
$$\mathcal{L}\{\cos kt\}(s) + i\,\mathcal{L}\{\sin kt\}(s) = \frac{s}{s^2+k^2} + i\,\frac{k}{s^2+k^2}$$

$f(t)$	$F(z)$
e^{at}	$\frac{1}{z-a}$
$\cos kt$	$\frac{z}{z^2+k^2}$
$\sin kt$	$\frac{k}{z^2+k^2}$

Theorem 1 If $f: [0,\infty) \to \mathbb{C}$ is of exponential order then $\exists!\, \sigma \in (-\infty, \infty)$ s.t.

$$\mathcal{L}\{f(t)\}(z) := \int_0^\infty e^{-zt} f(t)\,dt = F(z)$$

converges for $\mathrm{Re}\, z > \sigma$
diverges for $\mathrm{Re}\, z < \sigma$

And, on the half space $A := \{ z \in \mathbb{C} \text{ s.t. } \mathrm{Re}\, z > \sigma \}$

$F(z)$ is analytic, with

$$F'(z) = \int_0^\infty e^{-zt}(-tf(t))\,dt = -\mathcal{L}\{tf(t)\}(z)$$

idea of proof:

this is like the radius of convergence theorem:

σ can be defined to be the $\inf \{ \sigma_1 \text{ s.t. } \int_0^\infty e^{-\sigma_1 t} |f(t)|\,dt < \infty \}$

then, for $\mathrm{Re}\, z \geq \sigma + \varepsilon$,

$$F_N(z) := \int_0^N e^{-zt} f(t)\,dt$$

has $F_N'(z) = \int_0^N e^{-zt}(-t) f(t)\,dt$

$F_N \to F$ uniformly on $\{ \mathrm{Re}\, z \geq \sigma + \varepsilon \}$

$\Rightarrow F$ analytic and $F_N' \to F'$.

e.g. $f(t) = e^{at}$

$F(z) = \frac{1}{z-a}$

$-F'(z) = \mathcal{L}\{te^{at}\}$

$\quad\;\; = \frac{1}{(z-a)^2}$

Theorem 2 If $f(t), g(t)$ are continuous on $[0,\infty)$, and of exponential order, and if $\exists\, \sigma_1$ s.t. $F(s) = G(s)$ $\forall s \in \mathbb{R}, s \geq \sigma_1$ then $f(t) = g(t)$ $\forall t$. Laplace transform is 1-1!

(see page 466; analysis proof depends on Weierstrass approximation theorem)

Magic formula for inverse Laplace transform :

Let $F(z)$ analytic on $\mathbb{C}$ except for a finite number of isolated singularities

Let $\sigma \in \mathbb{R}$ s.t. F is analytic $\forall z$ s.t. $\operatorname{Re} z > \sigma$

Assume $\lim\limits_{z \to \infty} F(z) = 0$

Then
$$\mathcal{L}^{-1}\{F(z)\}(t) := f(t) = \sum_{\substack{z_j \text{ sing.} \\ \text{of } F}} \operatorname{Res}\left(e^{zt} F(z) ; z_j\right)$$

examples

$$F(z) = \frac{1}{(z-a)^2} \qquad f(t) = \operatorname{Res}\left(e^{zt} \frac{1}{(z-a)^2} ; a\right) = t e^{at}$$

$$\frac{e^{tz}}{(z-a)^2} = \frac{e^{ta} e^{t(z-a)}}{(z-a)^2} = e^{ta} \frac{(1 + t(z-a) + \dots)}{(z-a)^2}$$

$$F(z) = \frac{1}{z^2+a^2}$$

$$\begin{aligned} f(t) &= \operatorname{Res}\left(e^{zt} \frac{1}{z^2+a^2} ; ai\right) + \operatorname{Res}\left(e^{zt} \frac{1}{z^2+a^2} , -ai\right) \\ &= \frac{e^{ait}}{2ai} + \frac{e^{-ait}}{-2ai} \\ &= \frac{1}{a} \frac{1}{2i}\left(e^{ait} - e^{-ait}\right) \\ &= \frac{1}{a} \sin at \quad ! \end{aligned}$$

$$\frac{e^{zt}}{(z-ai)(z+ai)}$$

proof of inversion formula:

Let $f(t) := \sum_{z_j \text{ sing of } F} \operatorname{res}\left(e^{zt} F(z); z_j\right)$

assumptions:

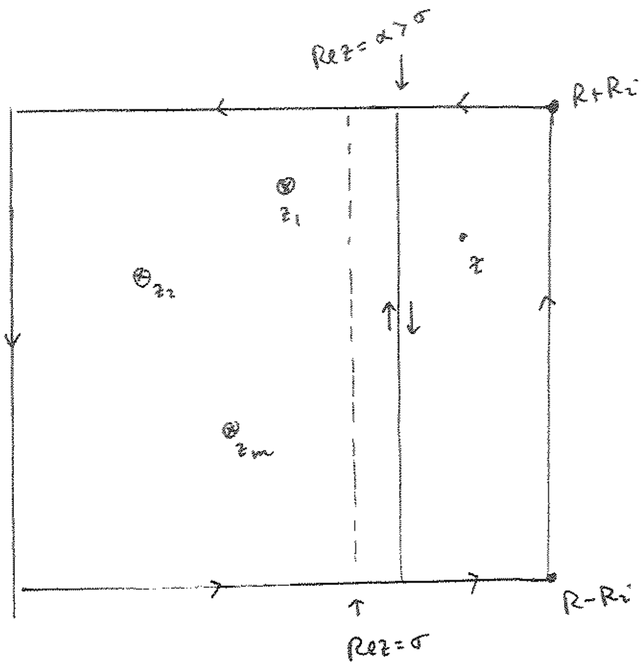
$\lim_{z \to \infty} F(z) = 0$

F analytic $\forall z$ s.t. $Re(z) > \sigma$

F has finite # of isolated singularities

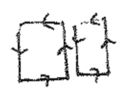
We show $\mathcal{L}\{f(t)\}(z) = F(z)$: (for $Re(z) > \sigma$)

$$\mathcal{L}\{f(t)\}(z) = \lim_{N \to \infty} \int_0^N e^{-zt} f(t)\,dt$$



Pick R large so that

γ encloses all singularities of F
$\tilde{\gamma}$ encloses z



$\gamma + \tilde{\gamma} = \Gamma$

Thus $f(t)$ as defined above

$$= \frac{1}{2\pi i} \oint_\gamma e^{\zeta t} F(\zeta)\,d\zeta$$

$$\Rightarrow \mathcal{L}\{f(t)\}(z) = \lim_{N \to \infty} \frac{1}{2\pi i} \int_0^N e^{-zt} \left(\int_\gamma e^{\zeta t} F(\zeta)\,d\zeta \right) dt$$

$$= \lim_{N \to \infty} \frac{1}{2\pi i} \int_0^N \left(\int_\gamma e^{t(\zeta - z)} F(\zeta) \right) d\zeta\,dt$$

$$= \lim_{N \to \infty} \frac{1}{2\pi i} \int_\gamma \underbrace{\left(\int_0^N e^{t(\zeta - z)} F(\zeta)\,dt \right)}_{\left. F(\zeta) \frac{e^{t(\zeta - z)}}{\zeta - z} \right]_{t=0}^{N}} d\zeta$$

← this is just change of order in double integrals, since after parameterization $d\zeta = \gamma'(\tau)\,d\tau$ and we're just switching $d\tau\,dt$ to $dt\,d\tau$

$$= \lim_{N \to \infty} \frac{1}{2\pi i} \int_\gamma F(\zeta) \left[\frac{e^{N(\zeta - z)}}{\zeta - z} - \frac{1}{\zeta - z} \right] d\zeta$$

$$= -\frac{1}{2\pi i} \int_\gamma \frac{F(\zeta)}{\zeta - z}\,d\zeta + \underbrace{\lim_{N \to \infty} \frac{1}{2\pi i} \int_\gamma F(\zeta) \frac{e^{N(\zeta - z)}}{\zeta - z}\,d\zeta}_{0}$$

$e^{N(\zeta - z)} \to 0$ uniformly as $N \to \infty$ since $Re(\zeta - z) < 0$ & $|e^{N(\zeta - z)}| = e^{N(Re(\zeta - z))}$

$$= \underbrace{\frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{F(\zeta)}{\zeta - z}\,d\zeta}_{F(z)} - \underbrace{\frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta)}{\zeta - z}\,d\zeta}_{0}$$

since $\gamma + \tilde{\gamma} = \Gamma$
$\gamma = \Gamma - \tilde{\gamma}$
$-\gamma = -\Gamma + \tilde{\gamma}$

$F(z)$: C.I.F.

$$\left| \quad \right| \le \frac{1}{2\pi} \cdot \text{length}(\Gamma) \cdot \max_{\zeta \in \Gamma} |F(\zeta)| \cdot \frac{1}{R} \le \frac{4}{\pi} \max_{\zeta \in \Gamma} |F(\zeta)| \to 0 \text{ as } R \to \infty$$

Fun curiosity:

Consider $F(z) = \dfrac{P(z)}{Q(z)}$ P, Q polynomials, $\deg P < \deg Q$

suppose Q has simple zeroes, i.e.

$Q(z) = a_n(z-z_1)(z-z_2)\cdots(z-z_n)$ z_j distinct

Then $\mathrm{Res}(F(z); z_j) = \dfrac{P(z_j)}{Q'(z_j)}$

$$\Rightarrow f(t) = \sum e^{z_j t} \frac{P(z_j)}{Q'(z_j)}$$

To find $\mathcal{L}^{-1}\{F(z)\}(t)$ in 2280/2250 you would've done partial fractions

$$\frac{P(z)}{Q(z)} = \sum_{j=1}^{n} \frac{c_j}{z-z_j}$$

$$\Rightarrow f(t) = \sum_{j=1}^{n} c_j e^{z_j t}$$

deduce partial fractions coeff's

$$c_j = \frac{P(z_j)}{Q'(z_j)}$$

(which you could also surely derive from elementary principles)

Ex $\dfrac{4z^3+7z+15}{z^3-z} = \dfrac{26}{2}\dfrac{1}{z-1} + \dfrac{15}{-1}\dfrac{1}{z} + \dfrac{4}{2}\dfrac{1}{z+1}$!

$P(z) = 4z^2+7z+15$

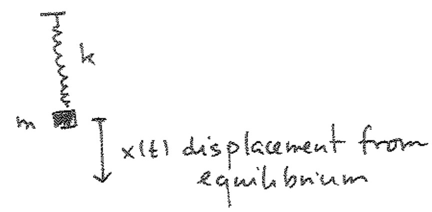
$Q(z) = z^3 - z = (z-1)z(z+1)$

$Q'(z) = 3z^2 - 1$

a reminder of how Laplace Transform is useful for solving linear DE's or linear systems of DE's.
(for people who avoided 2280/2250)

e.g. classical resonance problem

$$\begin{cases} x''(t) + \omega_0^2 x(t) = A\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$



Newton's 2nd Law:

$$m x'' = -kx + F(t)$$

Let $x(t)$ solve IVP. Write $X(z) = \mathcal{L}\{f(t)\}(z)$
See below to justify:

$$z^2 X(z) - z x_0 - v_0 + \omega_0^2 X(z) = A \frac{z}{z^2+\omega_0^2}$$

$$X(z)(z^2+\omega_0^2) = A\frac{z}{z^2+\omega_0^2} + x_0 z + v_0$$

$$X(z) = A\frac{z}{(z^2+\omega_0^2)^2} + x_0\frac{z}{z^2+\omega_0^2} + \frac{v_0}{z^2+\omega_0^2}$$

from Table deduce $\quad x(t) = A\frac{t}{2\omega_0}\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$

resonance

with no forcing ($A=0$) $\rightarrow$ simple harmonic motion
with forcing at natural frequency $\rightarrow$ resonance

$$\mathcal{L}\{f'(t)\}(z) = \int_0^\infty \underbrace{e^{-zt}}_{u}\underbrace{f'(t)dt}_{dv} = f(t)e^{-zt}\Big]_{t=0}^{\infty} - \int_0^\infty e^{-zt}(-z)f(t)dt$$

$du = e^{-zt}\,{-z}\,dt$

$$= 0 - f(0) + z\,\mathcal{L}\{f(t)\}(z)$$

So, $\mathcal{L}\{f''(t)\}(z) = -f'(0) + z\,\mathcal{L}\{f'(t)\}(z)$

$$= -f'(0) + z(-f(0) + z F(z))$$

$$= -f'(0) - z f(0) + z^2 F(z)$$