Math 4200
Monday 11/21 §4.4

We left off on Friday discussing how to explicitly sum

$$\sum_{\substack{n\in\mathbb{Z} \\ f \text{ not singular at } n}} f(n)$$

where $f(z)$ is any analytic function on $\mathbb{C}\setminus\{z_1, z_2 .. z_k\}$ which decays sufficiently fast at infinity.

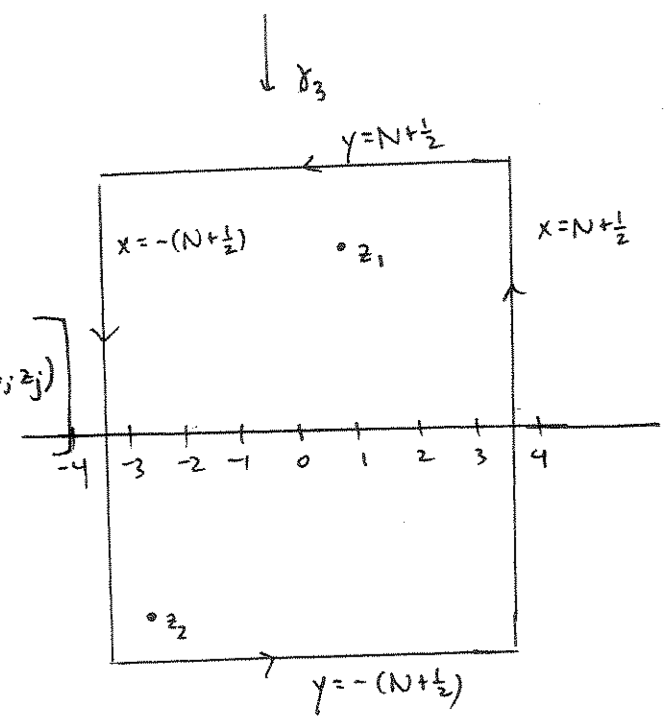
We use the auxillary function

$$\pi\cot\pi z = \pi\frac{\cos\pi z}{\sin\pi z}$$

which has simple poles at $z = \underline{n\in\mathbb{Z}}$, with residue $\underline{1}$.

We consider the Residue Theorem on rectangular cycles $\gamma_N = \partial R_N$:
And with integrand

$$f(z)\pi\cot\pi z$$



$$\int_{\gamma_N} f(z)\pi\cot\pi z\, dz$$

$$= 2\pi i\left[\sum_{\substack{n=-N \\ n \text{ not sing. pt of } f}}^{N} f(n) + \sum_{\substack{z_j\in R_N \\ \text{sing pt of } f}} Res(f(z)\pi\cot\pi z; z_j)\right]$$

Deduce

If $\lim_{N\to\infty}\int_{\gamma_N} f(z)\pi\cot\pi z\, dz = 0$

then $\lim_{N\to\infty}\sum_{\substack{n=-N \\ f(n) \text{ not sing.}}}^{N} f(n) = -\sum_{j=1}^{k} Res(f(z)\pi\cot\pi z; z_j).$

So, when can we deduce that the Contour integrals $\to 0$?

Lemma : $|\cot \pi z| \le 1$ on the vertical sides on γ_N
$|\cot \pi z| \to 1$ uniformly as $N \to \infty$, on horizontal sides of γ_N

proof: $$\frac{\cos \pi z}{\sin \pi z} = \frac{\frac{1}{2}(e^{i\pi z} + e^{-i\pi z})}{\frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})} = i\,\frac{e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}}{e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}}$$
$$= i\,\frac{e^{i\pi x}e^{-\pi y} + e^{-i\pi x}e^{\pi y}}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}}$$
$$= i\,\cancel{\frac{e^{i\pi x}e^{-\pi y}}{e^{i\pi x}e^{-\pi y}}}\left[\frac{1 + e^{-2i\pi x}e^{2\pi y}}{1 - e^{-2i\pi x}e^{2\pi y}}\right]$$
if $x = (N+\frac{1}{2})\pi$ this equals
$$i\left[\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right] \Rightarrow \text{vertical sides claim}$$

top: $y = N + \frac{1}{2}$
$$\left|\frac{1 + e^{-2\pi i x}e^{2\pi y}}{1 - e^{-2\pi i x}e^{2\pi y}}\right| \le \frac{e^{2\pi y} + 1}{e^{2\pi y} - 1} \to 1 \text{ as } y \to \infty.$$

bottom: $y = -(N+\frac{1}{2})$
$$\left|\frac{1 + e^{-2i\pi x}e^{2\pi y}}{1 - e^{-2i\pi x}e^{2\pi y}}\right| \le \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \to 1 \text{ as } y \to -\infty.$$

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Theorem If $f(z)$ is analytic on $\mathbb{C} \setminus \{z_1, z_2 .. z_k\}$
and $\exists\, M, R$ s.t.
$$|f(z)| \le \frac{M}{|z|} \quad \forall\, |z| \ge R$$
then $$\int_{\gamma_N} f(z)\pi \cot \pi z \, dz \to 0 \text{ as } N \to \infty$$

So $$\lim_{N\to\infty} \sum_{\substack{n=-N \\ n \text{ not sing pt}}}^{N} f(n) = -\sum_{j=1}^{k} \operatorname{Res}(f(z)\pi\cot\pi z; z_j)$$

pf: Consider the Laurent expansion for $f(z)$ in $|z| \ge R$.
It must be of the form
$$f(z) = \frac{b_1}{z} + \sum_{j=2}^{\infty} \frac{b_j}{z^j} = \frac{b_1}{z} + g(z), \quad g(z) \le \frac{C}{|z|^2} \text{ for } |z| \ge 2R$$

Thus $$\int_{\gamma_N} f(z)\pi\cot\pi z = \underbrace{\int_{\gamma_N} \underset{\text{odd}}{\frac{b_1}{z}}\ \underset{\text{odd}}{\pi\cot\pi z}\ \underset{\text{odd}}{dz}}_{\substack{=0 \text{ because Contour} \\ \text{is symmetric w.r.t.} \\ z \to -z}} + \int_{\gamma_N} g(z)\pi\cot\pi z\, dz$$

$$|\quad| \le \text{length}(\gamma_N)\cdot 2 \cdot \frac{\tilde{C}}{N^2}$$
$\to 0$ as $N \to \infty$.

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Examples

1. $f(z) = \frac{1}{z^2}$ only singular point is $z=0$

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^2} = -\mathrm{Res}\left(\frac{1}{z^2}\pi\cot\pi z; 0\right)$$

$$z\pi\cot\pi z = c_0 + c_2 z^2 + c_4 z^4 + \cdots \qquad \text{near } z=0$$

$$z\pi\cos\pi z = (c_0 + c_2 z^2 + c_4 z^4 + \cdots)\sin\pi z$$

$$z\pi\left[1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \cdots\right] =$$

$$(c_0 + c_2 z^2 + c_4 z^4 + \cdots)\left(\pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots\right)$$

for our residue we need c_2

z^1: $\quad c_0 = 1$

z^3: $\quad -\frac{1}{2}\pi^3 = -\frac{1}{6}\pi^3 + c_2\pi \quad \Rightarrow c_2 = -\frac{1}{3}\pi^2$

$$\Rightarrow 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{3}\pi^2, \qquad \boxed{\text{so } \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{6}\pi^2}$$

2. Using $f(z) = \frac{1}{z^4}, \frac{1}{z^6}, \frac{1}{z^{2M}}$ you get analogous magic formulas for $\sum_{n=1}^{\infty}\frac{1}{n^4}, \sum_{n=1}^{\infty}\frac{1}{n^6}$, etc. (i.e. the values of the Riemann-Zeta function at even positive integers).

3. Let $f(z) = \frac{1}{z-z_0}$, $z_0 \notin \mathbb{Z}$. only singular point is at z_0.

$$\Rightarrow \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{n-z_0} = -\mathrm{Res}\left(\frac{1}{z-z_0}\pi\cot\pi z; z_0\right) = -\pi\cot\pi z_0$$

So, (replacing z_0 with z & multiplying by -1)

$$\pi\cot\pi z = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z-n}$$

$$= \frac{1}{z} + \lim_{N\to\infty}\left[\sum_{n=1}^{N}\underbrace{\left[\left(\frac{1}{z-n}\right)+\frac{1}{n}\right]}_{=\frac{z}{n(z-n)}} + \sum_{n=1}^{N}\underbrace{\left[\frac{1}{z+n}-\frac{1}{n}\right]}_{=\frac{-z}{n(z+n)}}\right]$$

Thus

$$\pi\cot\pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{1}{z-n}+\frac{1}{n} + \sum_{n=1}^{\infty}\frac{1}{z+n}-\frac{1}{n}$$

these series converge uniformly absolutely on compact sets that avoid the integers, by comparison to $\sum\frac{1}{n^2}$ series.

Infinite partial fractions for meromorphic functions!

4) In your HW you'll prove

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

(notice pole locations & orders agree on ~~both~~ both sides.

[The proof is one line from the previous page, if you can think of it.]

These are examples of a general theory for infinite sum partial fraction expansions for meromorphic functions, i.e. functions analytic on $\mathbb{C}$ except for an at most countable set $\{z_k\}$ of isolated pole singularities. (as opposed to essential singularities)

The Mittag-Leffler Theorem says you can basically create an infinite sum analytic function with prescribed isolated poles, and with prescribed negative power Laurent series terms at those poles.

Conversely, for a given meromorphic function, one can try to write down a Mittag-Leffler expansion which gives the same negative power Laurent series at each singular point.

- for example, for $\frac{\pi^2}{\sin^2 \pi z} \xrightarrow{?} \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ would be a good first guess.

Then, the difference of the two functions will have removable singularities at all previous singular points, so will be entire. After perhaps adding another entire function to this difference, one might hope to have a bounded entire function $\Rightarrow$ constant by Liouville's Thm!

e.g. $\frac{\pi^2}{\sin^2 \pi z} - \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ extends to be entire.

- is periodic in the x (real) direction, with period $p = 1$.
- each ~~term~~ of the two fcns in the difference $\to 0$ uniformly as $|y| \to \infty$

$\Rightarrow$ the difference is bounded
$\Rightarrow$ const. Letting $|y| \to \infty \Rightarrow$ const $= 0$.

[alternate proof of identity 4].

There is an analogous theory for infinite product analytic functions
[based on whether the logarithms of the partial products, which are partial sums for infinite series, converge].

There are equally magical infinite product identities. - see e.g. Chapter 7 or identities related to the Riemann-Zeta function