Math 4200
Wed 11/2

§3.3

HW: postpone §3.3 #9, 17, 20 until next week
So for Friday, finish
3.2 5c, 7, 13, 14, 18, 19, 20
3.3 1ab, 4, 6, 8, 13, 15, 18, 19

We've proved ② ⇒ ① of the Theorem below.

Today: ① ⇒ ② ; ③ ; "residues"; examples

Laurent series Theorem

Let $A = \{ z \mid r_1 < |z-z_0| < r_2 \}$ $\quad r_1 \geq 0,\ r_1 < r_2$, be an open annulus

Then ① and ② are equivalent:

① $f: A \to \mathbb{C}$ is analytic

② $f(z)$ has power series expansion (with positive and negative powers):

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$$

$$:= S_1(z) + S_2(z)$$

Here $S_1(z)$ converges for $|z-z_0| < r_2$ (hence uniformly absolutely for $|z-z_0| \leq r_2 - \varepsilon$, $\varepsilon > 0$)

and $S_2(z)$ converges for $|z-z_0| > r_1$ (hence uniformly absolutely for $|z-z_0| \geq r_1 + \varepsilon$)

Also

③ the coefficients a_n, b_m are uniquely determined by f

Specifically, if γ is any p.w. C^1 curve with image$(\gamma) \subset A$ and $I(\gamma; z_0) = 1$ (e.g. $\gamma(t) = z_0 + re^{it}$ $\ 0 \leq t \leq 2\pi$, $r_1 < r < r_2$)

then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$b_m = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\zeta - z_0)^{m-1}\, d\zeta$$

In particular, $\int_\gamma f(\zeta)\, d\zeta = 2\pi i b_1$. For this reason, b_1 is called the residue of f at z_0

1 $\Rightarrow$ 2 : Let $\varepsilon > 0$.

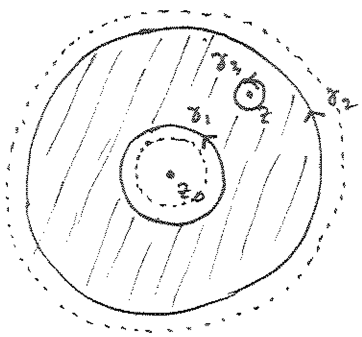
Let $A_\varepsilon = \{ z \in A \mid r_1 + \varepsilon \le |z - z_0| \le r_2 - \varepsilon \}$

Let γ_1 = circle of radius $r_1 + \varepsilon/2$ about z_0

γ_2 = circle of radius $r_2 - \varepsilon/2$ about z_0

Let $z \in A_\varepsilon$, γ_3 = circle of radius $\varepsilon/4$ about z

} all circles oriented c.c., as usual

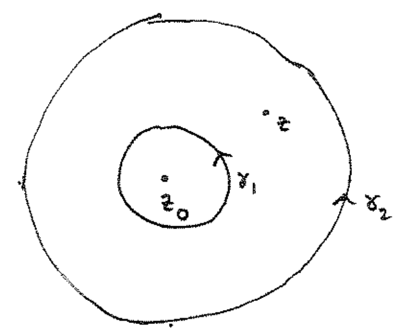


$\gamma_2 - \gamma_1 - \gamma_3$ is the oriented boundary of a region which is a disk with two holes, and this region is compactly contained within A.

Green's Thm version of Cauchy's formula

$$\Rightarrow 0 = \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \underbrace{\int_{\gamma_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{f(z)\, 2\pi i\, \underbrace{(I(\gamma_3; z))}_{1}}$$

$$\Rightarrow \boxed{f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta}$$



$$\frac{1}{2\pi i} \int_{|\zeta - z_0| = r_2 - \varepsilon/2} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$\frac{f(\zeta)}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)}\, d\zeta$$

"w", $|w| \le \frac{1-\varepsilon}{1 - \varepsilon/2} =: \mu < 1$

$$\frac{1}{2\pi i} \int_{\gamma_2} \sum_{n=0}^{\infty} \underbrace{\frac{f(\zeta)}{(\zeta - z_0)} \left(\frac{z - z_0}{\zeta - z_0}\right)^n}\, d\zeta$$

$|\ \%\ | \le \frac{M}{r_2 - \varepsilon/2} \mu^n$

unif (abs) conv $\Rightarrow$ interchange of sum & integral is valid

$$\underbrace{\sum_{n=0}^{\infty} (z - z_0)^n \underbrace{\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta}_{:= a_n}}_{= S_1(z)}$$

$$+ \frac{1}{2\pi i} \int_{|\zeta - z_0| = r_1 + \varepsilon/2} \frac{f(\zeta)}{(z - z_0) - (\zeta - z_0)}\, d\zeta$$

$$\frac{f(\zeta)}{(z - z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)}\, d\zeta$$

"w", $|w| \le \frac{r_1 + \varepsilon/2}{r_1 + \varepsilon} = \mu < 1$.

$$\frac{1}{2\pi i} \int_{|\zeta - z_0| = r_1 + \varepsilon} \sum_{k=0}^{\infty} \frac{f(\zeta)}{z - z_0} \left(\frac{\zeta - z_0}{z - z_0}\right)^k d\zeta$$

Let $m = k+1$

$m = 1 .. \infty$,

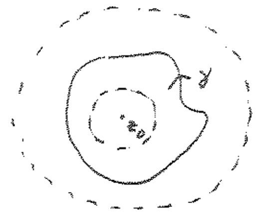
interchange sum & integral

$$\underbrace{\sum_{m=1}^{\infty} \frac{1}{(z - z_0)^m} \underbrace{\frac{1}{2\pi i} \int_{\gamma_1} f(\zeta)(\zeta - z_0)^{m-1}\, d\zeta}_{:= b_m}}_{S_2(z)}$$

③ Uniqueness.

let γ p.w. C^1, closed, $I(\gamma; z_0) = +1$

Image$(\gamma) \subset A$



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$$

$$= S_1(z) + S_2(z)$$

- $$\int_\gamma f(\zeta)\,d\zeta = \int_\gamma S_1(\zeta)\,d\zeta + \int_\gamma S_2(\zeta)\,d\zeta$$

$$= \int_\gamma \sum_{n=0}^{\infty} a_n (\zeta-z_0)^n\,d\zeta + \int_\gamma \sum_{m=1}^{\infty} \frac{b_m}{(\zeta-z_0)^m}\,d\zeta$$

$$= \sum_{n=0}^{\infty} \underbrace{\int_\gamma a_n(\zeta-z_0)^n\,d\zeta}_{=\,0} + \sum_{m=1}^{\infty} \underbrace{\int_\gamma \frac{b_m}{(z-z_0)^m}\,d\zeta}_{=\,0}$$

(the usual justification)

since $\int (\zeta-z_0)^n\,d\zeta = \frac{1}{n+1}(\zeta-z_0)^{n+1}$

except for $m=1$, where result is $2\pi i\, b_1$

$$= 0 + 2\pi i b_1$$

- similarly, if $n_1 \geq 0$ (integer),

$$\int_\gamma \frac{f(\zeta)}{(\zeta-z_0)^{n_1+1}}\,d\zeta = \int_\gamma \sum_{n=0}^{\infty} a_n \frac{(\zeta-z_0)^n}{(\zeta-z_0)^{n_1+1}}\,d\zeta + \int_\gamma \sum_{m=1}^{\infty} \frac{b_m}{(\zeta-z_0)^m(\zeta-z_0)^{n_1+1}}\,d\zeta$$

$$= 2\pi i\, a_{n_1} + 0$$

& if $m_1 \geq 1$

$$\int_\gamma f(\zeta)(\zeta-z_0)^{m_1-1}\,d\zeta = 0 + \int_\gamma \sum_{m=1}^{\infty} \frac{b_m}{(\zeta-z_0)^m}(\zeta-z_0)^{m_1-1}\,d\zeta$$

$$= 2\pi i\, b_{m_1}$$

Examples On Monday

we did $\frac{1}{(z-1)(z+2)} = \frac{1}{3}\left(\frac{1}{z-1} - \frac{1}{z+2}\right)$

$z_0 = 0$

in

(a) $|z|<1$ Taylor

(b) $1<|z|<2$ Laurent

(c) $|z|>2$ Laurent

Exercise 1 : Find the Laurent series for $ze^{\frac{1}{z}}$ in $\mathbb{C}\setminus\{0\}$,
and use it to deduce

$$\int_\gamma z e^{1/z}\, dz$$

if γ is a closed curve with $I(\gamma;0)=1$. What is the residue of $ze^{\frac{1}{z}}$ @ $z=0$?

Exercise 2 : Let $f(z) = \frac{e^z}{\sin z}$, $0<|z|<\pi$.

Find the Laurent series, at least the first four non-zero terms.

Hint: First deduce

$z\,f(z)$

is analytic. Then use Taylors series thm about multiplying power series