Math 4200
Friday 11/18
§ 4.3 – 4.4

Before the midterm exam we discussed how to use clever contours to evaluate real variables integrals. Today and Monday we'll continue this discussion, and also discuss a technique for computing certain infinite sums.

Last Friday:

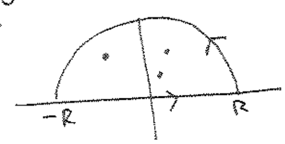
1 $\displaystyle\int_0^{2\pi} f(\cos\Theta, \sin\Theta)\, d\Theta \longrightarrow \oint_{|z|=1} f\left(\tfrac{1}{2}(z+\tfrac{1}{z}), \tfrac{1}{2i}(z-\tfrac{1}{z})\right)\frac{dz}{iz} = 2\pi i \sum_k \operatorname{Res}(g(z); z_k)$

$g(z) = \frac{1}{iz} f\left(\tfrac{1}{2}(z+\tfrac{1}{z}), \tfrac{1}{2i}(z-\tfrac{1}{z})\right)$

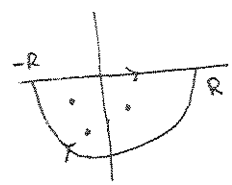
z_k = singularities inside $|z|=1$.

2 $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ if you can carry out a limiting procedure from

$\displaystyle\oint_{\gamma_R} f(z)\,dz$



or

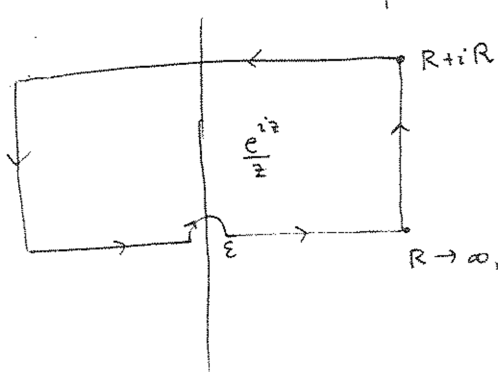


$\displaystyle\oint_{\gamma_R} f(z)\,dz$

a variation on 2

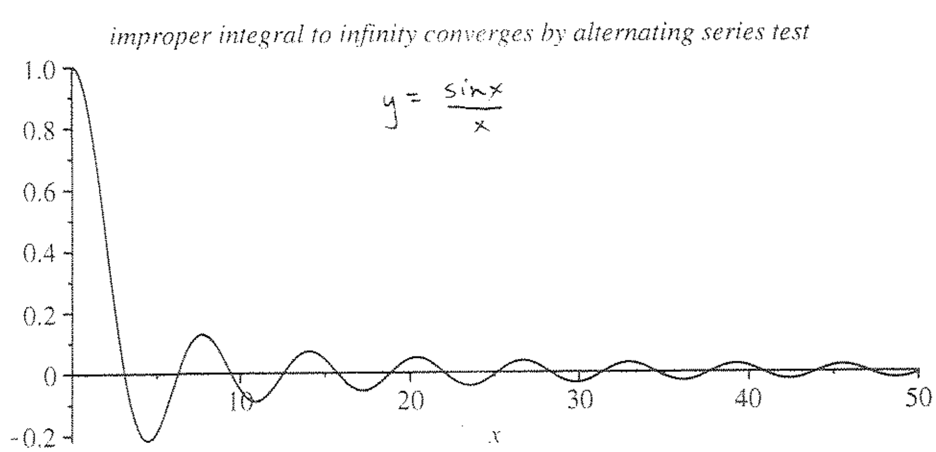
Exercise 1 Prove $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx = \pi/2$

using $\displaystyle\int_{\gamma_{\varepsilon,R}} \frac{e^{iz}}{z}\,dz$



$R\to\infty$, before $\varepsilon\to 0$

First compute $\displaystyle\int_{\varepsilon}^{\infty} \frac{\sin x}{x}\,dx$, then let $\varepsilon\to 0$.

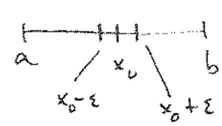


note, $\frac{e^{iz}}{z}$ has a singularity @ $z=0$, even though $\frac{\sin x}{x}$ is continuous @ $x=0$.

there is a more general class of integrals, called Principal Value (or PV) integrals, that one can compute, even when the actual integral doesn't exist.

Def. If f is continuous on $[a,b]$ except at $x_0 \in (a,b)$, then

$$PV\left(\int_a^b f(x)\,dx\right) := \lim_{\varepsilon \to 0}\left[\int_a^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^b f(x)\,dx\right],$$ provided this limit exists.

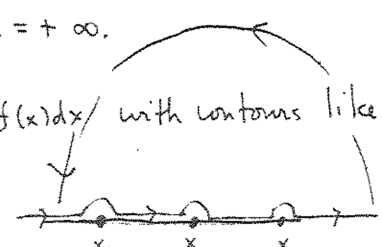


e.g. $PV\left(\int_{-1}^{2} \frac{1}{x}\,dx\right) = \ln 2$

even though $\int_{-1}^{0} \frac{1}{x}\,dx = -\infty$ and $\int_0^1 \frac{1}{x}\,dx = +\infty$.

Using principle value idea, one can compute $PV\int_{-\infty}^{\infty} f(x)\,dx$ with contours like

(prop 4.3.11 in text.)



Magic formulas for infinite series (§4.4)

Consider $f(z)\pi\cot\pi z$, where $f(z)$ is defined on $\mathbb{C}\setminus\{z_1, z_2, \ldots z_k\}$.
& analytic

Example $\frac{1}{z^2}\pi\cot\pi z = g(z)$, i.e. $f(z) = \frac{1}{z^2}$.

$\pi\cot\pi z = \frac{\pi\cos\pi z}{\sin\pi z}$ has simple poles at each integer

so if $f(n)\neq 0$, $\operatorname{Res}\left(\frac{f(z)\pi\cos\pi z}{\sin\pi z}; n\right) = \underbrace{\frac{f(n)\,\pi\cos\pi n}{\pi\cos\pi n}}_{\frac{d}{dz}\sin\pi z = \pi\cos\pi z} = \boxed{f(n)}$

for $\boxed{f(z) = \frac{1}{z^2}}$ we get $\operatorname{Res}\left(\frac{1}{z^2}\pi\cot\pi z; n\right) = \boxed{\frac{1}{n^2}}$

for $n\neq 0$

for $n=0$ the pole at $z=0$
of $\frac{1}{z^2}\pi\cot\pi z$ is of order 3
We can use technology to shortcut finding the Laurent series of $\pi\cot\pi z$:
Maple says:

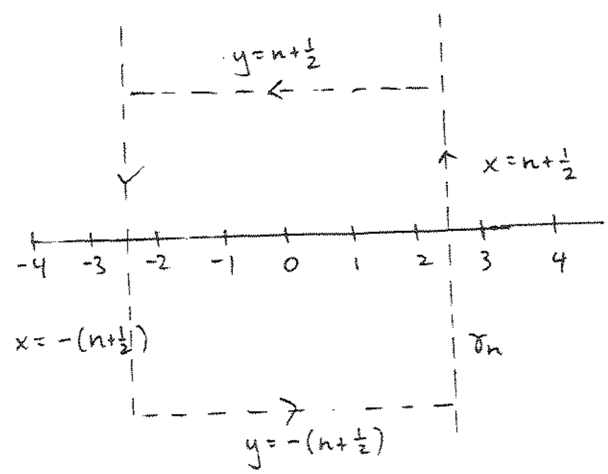
> series(Pi·cot(Pi·z), z, 12);

$$z^{-1} - \frac{1}{3}\pi^2 z - \frac{1}{45}\pi^4 z^3 - \frac{2}{945}\pi^6 z^5 - \frac{1}{4725}\pi^8 z^7 - \frac{2}{93555}\pi^{10} z^9 - \frac{1382}{638512875}\pi^{12} z^{11} + O(z^{12})$$

$\Rightarrow \operatorname{Res}\left(\frac{1}{z^2}\pi\cot\pi z; 0\right) = \boxed{-\frac{1}{3}\pi^2}$

Now consider

$\int_{\gamma_n} f(z)\pi\cot\pi z\, dz$: γ_n traces a square:



This square is carefully chosen so that $|\cot\pi z| \leq 2$ on γ_n

Thus, if $|f(z)| \leq \frac{C}{|z|^2}$ for $|z|$ large,

$\left|\int_{\gamma_n} f(z)\pi\cot\pi z\, dz\right| \leq 4(2n+1)\cdot\pi\cdot 2\cdot\frac{C}{n^2} \to 0$ as $n\to\infty$.

Residue Thm $\Rightarrow \lim_{n\to\infty} \sum_{j=-n}^{n} f(j) = -\sum_{z_k} \operatorname{Res}\left(f(z)\pi\cot\pi z; z_k\right)$

j not a singular point of $f(z)$ — z_k singular points of f

Exercise 3 Use the previous page to deduce

$$\sum_{n=1}^{\infty} \frac{1}{n^2} =$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} =$$

Why can't you deduce a value for $\sum_{n=1}^{\infty} \frac{1}{n^3}$?

Exercise 4 Prove that $|\pi \cot \pi z| \leq 2$ (for n large) on γ_n, which was needed to make the contour integral $\to 0$ as $n \to \infty$.