Math 4200-1
Fri 11/11

Hw for 11/23 (Wed of Thanksgiving Week)
4.3 1, 2, 4, 6, 10, 14, 17, 20ab
4.4 1, 2, 3, 4, 5, 8, 9

§4.3 Real variables definite integrals via Residue Theorem magic.

① $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ where f is any rational function of $\cos\theta, \sin\theta$ (in fact, if $f(z,w)$ is analytic in z and in w, you can use this method).

this is a disguised contour integral around the unit circle.

if $z = e^{i\theta}$ $\quad 0 \le \theta \le 2\pi$

$\Rightarrow \frac{1}{z} = e^{-i\theta}$

$\Rightarrow \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \boxed{\frac{1}{2}\left(z + \frac{1}{z}\right)}$

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \boxed{\frac{1}{2i}\left(z - \frac{1}{z}\right)}$

$dz = i e^{i\theta} d\theta$ so $\boxed{d\theta = \frac{dz}{iz}}$

Thus $$\oint_{|z|=1} f\left(\tfrac{1}{2}\left(z+\tfrac{1}{z}\right), \tfrac{1}{2i}\left(z-\tfrac{1}{z}\right)\right) \frac{dz}{iz} = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

when you parameterize unit circle by $z = e^{i\theta}$

$$\| $$

$$2\pi i \sum \text{residues of } \left[f\left(\tfrac{1}{2}\left(z+\tfrac{1}{z}\right), \tfrac{1}{2i}\left(z-\tfrac{1}{z}\right)\right)\tfrac{1}{iz}\right] \text{ inside the unit circle.}$$

Example (one you know). ~ (since $\cos^2\theta = \frac{1+\cos 2\theta}{2}$)

$\int_0^{2\pi} \cos^2\theta \, d\theta$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$\cos^2\theta = \frac{1}{4}\left(z^2 + 2 + \frac{1}{z^2}\right)$

$$\| $$

$$\oint_{|z|=1} \frac{1}{4}\left(z^2 + 2 + \frac{1}{z^2}\right) \frac{1}{iz}\, dz$$

$$\| $$

$$2\pi i \left(\text{Res}(\%; 0)\right)$$

$$\| $$

$$2\pi i \, \frac{1}{2i} = \boxed{\pi} \checkmark$$

Exercise 1 $\displaystyle\int_0^{\pi} \cos^4\theta \, d\theta = \frac{3}{8}\pi$

Exercise 2 (Relates to Poisson integral formula)

$$\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2\pi}{|a^2-1|} \qquad (a \text{ real}, \ a \neq \pm 1)$$

note: denom $= (1-a\cos\theta)^2 + a^2\sin^2\theta \neq 0$ for $a \neq \pm 1$.

note: for simple pole @ z_0

$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

2 Integrals along the real line for rational functions

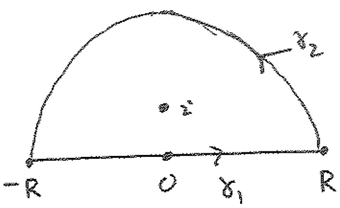
Example (one you know)

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \arctan(x)\Big]_0^\infty = \pi/2 - 0 = \pi/2$$

via contour integral.

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{1+x^2}\,dx$$

consider γ_R:



$\gamma_R = \gamma_1 + \gamma_2$

$$\boxed{\int_{\gamma_R} \frac{1}{1+z^2}\,dz = \int_{\gamma_1} \frac{dz}{1+z^2} + \int_{\gamma_2} \frac{dz}{1+z^2}}$$

$(R>1)$

$$\int_{\gamma_R} \frac{1}{1+z^2}\,dz = 2\pi i\left(\mathrm{Res}\left(\frac{1}{1+z^2}; i\right)\right) = \pi$$

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$$

so $\mathrm{Res} = \frac{1}{(i+i)} = \frac{1}{2i}$

$$\int_{\gamma_1} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2}$$

$$\left|\int_{\gamma_2} \frac{dz}{1+z^2}\right| \le \int_{\gamma_2} \frac{1}{R^2-1}\,|dz| = \frac{\pi R}{R^2-1} \to 0 \quad \text{as } R\to\infty$$

$\lim_{R\to\infty}$ $\boxed{}$

$$\Rightarrow \pi = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx, \quad \text{so } \int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

this method will work for any rational function (quotient of polys)

$$f(x) = \frac{p(x)}{q(x)}$$

provided $q(x)$ has no real zeroes and $\text{degree}(q) \geq \text{degree}(p) + 2$

in case you wish to integrate

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$$

(and in case you can find the zeroes of q.)

3 More complicated integrals along the real line

there are lots of interesting ones, often requiring clever choice of function or contour.

<u>Example</u> $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx$ $a \neq 0$, real, $a>0$ (without loss of generality)

first try $f(z) = \dfrac{\cos az}{1+z^2} = \dfrac{\cos az}{(z-i)(z+i)}$. This try will fail:

let $R \to \infty$.

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{-R}^{R} \frac{\cos ax}{1+x^2}\,dx + \int_{\gamma_2} \frac{\cos az}{1+z^2}\,dz$$

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = 2\pi i\,(\mathrm{Res}(f(z); i)) = 2\pi i\left(\frac{\cos ai}{i+i}\right) = \pi \cosh a$$

$$\int_{-R}^{R} \frac{\cos ax}{1+x^2}\,dx \to \int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx$$

$$\left|\int_{\gamma_2} \frac{\cos az}{1+z^2}\,dz\right| \leq \int_{\gamma_2} \frac{|\cos az|}{R^2-1}\,|dz|$$

$$\cos a(x+iy) = \tfrac{1}{2}\left(e^{ia(x+iy)} + e^{-ia(x+iy)}\right) = \tfrac{1}{2}\left(e^{-ay}e^{iax} + e^{ay}e^{-iax}\right)$$

trouble in lower half plane $|\ \ | = e^{-ay}$

trouble in upper half plane $|\ \ | = e^{ay}$

<u>Exercise 3</u> Compute the correct value of $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\,dx$

by considering $\mathrm{Re} \displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,dx$

i.e. $f(z) = \dfrac{e^{iaz}}{1+z^2}$

correct ans: πe^{-a}