Math 4200
Final exam review sheet
December 9, 2011

   The final exam is this Tuesday, December 13, in our classroom JTB 120, from 10:30–12:30 (but I won't kick anyone out until 1:00). The exam is comprehensive, covering chapters 1–5.2 . On the exam you will be asked to pick approximately six of ten problems. At least three of the exam problems will be chosen directly from our first two midterms (which are posted, along with solutions, on our home page), and at least two more will be favorites from homework. Several problems will ask you to prove key theorems from complex analysis. If you want to see my actual exam from fall 2002, follow the link from our lecture page.

   If you wish to ask me study–related questions, come by my office (maybe call first, 581–7318 to see if I'm in), or send an email so that we can set up a time.

   I think of basic complex analysis as following from (only) several key ideas, and the core of these ideas is what I hope you carry away from the course, along with the correct impression that interesting complex analysis magic appears in a lot of different areas of higher math. Here are the core ideas:

(0) Complex number algebra and geometry: addition, multiplication, conjugation, inverse, Euler's formula and the polar form of complex numbers. It is essential to understand this algebra and geometry, in order to understand the basic analytic functions. (And this is the starting point in the complex discrete dynamical systems for Julia sets and Mandelbrot sets, as we see Friday.)

(1) $f(z) = f(x + i y) = u(x, y) + i v(x, y)$ is complex differentiable at z_0 (the limit definition of complex derivative) iff $F(x, y) = (u(x, y), v(x, y))$ is real differentiable, with derivative matrix a rotation dilation. This circle of ideas includes the Cauchy Riemann equations, the chain rule for $f(\gamma(t))$, where $f(z)$ is analytic and $\gamma(t)$ is a parametric curve. It also includes the calculus of complex differentiation and the inverse function theorem for analytic functions, and leads to the geometry of conformal transformations.

(2) Conformal transformations (bijections) between specified open, connected domains. This includes the zoo of transformations we studied in chapter 1 (all the functions on your scientific calculator), including the concepts of branch points and branch cuts, combined with the fractional linear transformations and RMT in chapter 5.1–5.2. (This direction leads to applications in geometry and partial differential equations, although this year nobody chose these topics for project investigation.)

(3) Contour integrals of analytic functions,

$$\int_{\gamma} f(z) \, dz \, .$$

how to compute by parameterization; the real and imaginary part are each real line integrals; why such integrals equal zero when γ is the oriented boundary of a domain (Green's Theorem and CR equations!... Green's Theorem also yields alternate proofs of the Cauchy integral formula and the residue theorem in "simple" cases.)

(4) The circle of theorems related to the Cauchy integral formula:
(4a) f analytic in a disk ==> rectangle theorem in disk ==> antiderivative in disk.
local antiderivatives implies the deformation theorem for two homotopic closed curves in an domain A (which says that the contour integrals over the two curves must agree). In particular, for a simply connected domain, contour integrals of analytic functions over closed curves must be zero, implying global antiderivatives exist.

(4b) definition of index, computation via a contour integral, and the derivation of the Cauchy integral formula, using (4a).
(4c) CIF for derivatives. Liouville. FTA. mean value and maximum modulus principle properties for analytic functions.

(5) Power series and Laurent series.
(5a) radius of convergence for power series (and uniform absolute convergence inside), and complementary result for power series in negative powers. Annulus of convergence consequence for a series with positive and negative powers. The fact that the resulting sums are analytic, using the Weierstrass M test and the fact that uniform limits of analytic functions are analytic, with limit derivative equal the limit of the derivatives. (uses Morera, CIF!).
(5b) derivation and uniqueness of power series and Laurent series for analytic functions in disks and annuli. (The amazing uses of geometric series.)
(5c) uniqueness of analytic extension ... based on the theorem that zeroes of analytic functions are isolated, so if two analytic functions agree on a sequence converging to a point not in the sequence, then they agree on their common domain.
(5d) radius of convergence for power and Laurent series, based on explicit convergence tests or on domain of analyticity of a function.
(5e) finding power and Laurent series for given functions, e.g. method of equating coefficients in products or quotients...and you should know the key power series (trig, geometric, power etc.) if you hope to get started!

(6) Residue calculus
(6a) computing residues. (I will provide you with the text's table of residue computation tools, but we've discussed the key ideas which often provide shortcuts and more effective ways to compute these.)
(6b) statement and proof of the residue theorem.
(6c) computing integrals with the residue theorem, or with some limiting process which starts with the residue theorem. (I will provide you with the table on page 296 of the text, but you will need to justify the usage of any such formula with estimates, when appropriate.)
(6d) summing series with residue calculus. (See homework and class notes for representative examples.. ..We just touched on magic formulas for "infinite partial fractions", and as we saw in the Riemann Zeta function presentation there are related magic product formulas with which one can often express functions with inifinitely many zeroes as infinite products.)

(7) Applications to PDE's: We really only talked about harmonic functions, and did not do much on this topic this year – In our text this is covered pretty well in section 5.3, although we did not cover this section and you won't be responsible for it. But you should know the relationship between analytic functions and harmonic functions: harmonic conjugates in simply connected domains, mean value and maximum/minimum principles, harmonic composed with analytic is harmonic, which lets you find harmonic functions in one domain by finding harmonic functions in another.