Math 4200
exam 2 sol'ns

1. $f(z) = \frac{2}{z} + \frac{3}{z-1}$

a) $0 < |z| < 1$: $\frac{3}{z-1} = \frac{-3}{1-z} = -3\sum_{n=0}^{\infty} z^n$ geometric series; $|z|<1$

so $f(z) = \frac{2}{z} - 3\sum_{n=0}^{\infty} z^n$

b) $|z| > 1$: geometric series; $|\frac{1}{z}|<1$

$$\frac{3}{z-1} = \frac{3}{z}\,\frac{1}{1-\frac{1}{z}} = \frac{3}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}$$

so $f(z) = \frac{2}{z} + \frac{3}{z} + \sum_{m=2}^{\infty}\frac{3}{z^m}$

$= \frac{5}{z} + \sum_{m=2}^{\infty}\frac{3}{z^m}$.

c) $\oint_{|z|=2} \frac{2}{z} + \frac{3}{z-1}\,dz$

$= 2\pi i\,(\mathrm{Res}(f;0) + \mathrm{Res}(f;1))$

$= 2\pi i\,(2+3) = 10\pi i$

Alternately we could have integrated $\frac{5}{z} + \sum_{m=2}^{\infty}\frac{3}{z^m}$ term by term (since the series converges uniformly on $|z|=2$. $\int_{|z|=2}\frac{5}{z}\,dz = 5\cdot 2\pi i = 10\pi i$

the other terms integrate to zero since they all have anti-derivatives

2a) Liouville's Theorem: Let $f(z)$ entire and $|f(z)| \le M \ \forall z \in \mathbb{C}$.
Then f is constant.

proof. Let $\gamma = \{z \text{ s.t. } |z-z_0| = R\}$.

Then $$f'(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^2}\,dz$$

$$\Rightarrow |f'(z_0)| \le \frac{1}{2\pi}\int_\gamma \frac{|f(z)|}{|z-z_0|^2}|dz| \le \frac{1}{2\pi}\int_{|z|=R} \frac{M}{R^2}|dz|$$

$$= \frac{1}{\cancel{2\pi}}\frac{M}{R^2}\cancel{2\pi} R = \frac{M}{R}.$$

Letting $R \to \infty$ we deduce

$$|f'(z_0)| = 0.$$

Thus $f'(z_0) = 0 \ \forall z_0 \in \mathbb{C}$

$\Rightarrow f(z) = c \in \mathbb{C}$.

2b) FTA: Let $p_n(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ be any (monic) polynomial of degree n. Then $p_n(z)$ factors completely, into linear factors
$p_n(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$.

proof Let $n=1$. Then $p_1(z) = z + a_0 = (z-(-a_0))$

induction step: If the theorem is true for $n-1$, then it suffices to find a single root z_1 for $p_n(z)$, since $p_n(z_1)$ implies $p_n(z) = (z-z_1)\,p_{n-1}(z)$ (since the remainder of $\frac{p_n(z)}{z-z_1}$ is exactly $p_n(z_1)$)
and by induction hypothesis p_{n-1} factors completely.

If, for some $p_n(z)$ with $n \geq 1$, there is no root, then

$\frac{1}{p_n(z)}$ is entire.

Also, $$\frac{1}{p_n(z)} = \underbrace{\frac{1}{z^n}}_{\lim_{z\to\infty}(\ )=0}\ \underbrace{\frac{1}{\left(1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right)}}_{\lim_{z\to\infty}(\ )=1}$$

So $\lim_{n\to\infty}\left|\frac{1}{p_n(z)}\right| = 0$. Thus $\left|\frac{1}{p_n(z)}\right|$ is bounded on $\mathbb{C}$, since $\exists R$ s.t. $|z| > R \Rightarrow \left|\frac{1}{p_n(z)}\right| \le 1$, and $\left|\frac{1}{p_n(z)}\right|$ is cont on the closed disk $|z| \le R \Rightarrow$ bounded as well. Thus $\frac{1}{p_n(z)} = \text{const}$

by Liouville.
Thus $p(z) = \text{const}$ $*$.

3a $\displaystyle\int_{|z|=8} \frac{\sin 2z}{z^2}\,dz = \int_{|z|=\infty} \frac{2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots}{z^2}\,dz$

$\displaystyle = \int_{|z|=8} \frac{2}{z} - \frac{2^3}{3!}z + \frac{2^5}{5!}z^3 - \cdots = \int_{|z|=8} \frac{2}{z}\,dz = 2\cdot 2\pi i = \boxed{4\pi i}$

(or, use $\mathrm{Res}\left(\frac{\sin 2z}{z^2}; 0\right) = 2$).

3b. $\displaystyle\int_{|z|=8} z^2 e^{\frac{1}{z}}\,dz = \int_{|z|=8} z^2\left[1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots\right]dz$

$\displaystyle = \int_{|z|=8} z^2 + z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots dz$

interchang integral & sum because conv is uniform, all terms integrate to zero excep the $\frac{1}{z}$ term

$\displaystyle\Rightarrow \int_{|z|=8} z^2 e^{\frac{1}{z}}\,dz = 2\pi i\left(\frac{1}{3!}\right) = \boxed{\frac{\pi i}{3}}$

(or use Residue thm)

3c. $\displaystyle\int_{|z|=8} \frac{z^2}{\sin 2z}\,dz$

$z=0$ is removable singularity

other poles inside $|z|=8$ are simple,

@ $2z = \pm\pi$, $2z = \pm 2\pi$, $2z = \pm 3\pi$, $2z = \pm 4\pi$, $2z = \pm 5\pi$

i.e. $z = \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \pm\frac{5}{2}\pi$

$\displaystyle\mathrm{Res}\left(\frac{z^2}{\sin 2z}; z_0\right) = \frac{z_0^2}{2\cos 2z_0}$ @ each simple pole.

$\displaystyle\Rightarrow \int_{|z|=8} \frac{z^2}{\sin 2z}\,dz = 2\pi i\left[\frac{2\left(\frac{\pi}{2}\right)^2}{-2} + \frac{2\pi^2}{+2} + \frac{2\left(\frac{3}{2}\pi\right)^2}{-2} + \frac{2(2\pi)^2}{+2} + \frac{2\left(\frac{5}{2}\pi\right)^2}{-2}\right]$

$\displaystyle = 2\pi i\,\pi^2\left[-\frac{1}{4} + 1 - \frac{9}{4} + 4 - \frac{25}{4}\right]$ $\qquad -\frac{35}{4} + \frac{20}{4} = -\frac{15}{4}$

$\displaystyle\boxed{= -\frac{15}{2}\pi^3 i}$

4. Residue Thm version 1: let $f : A \setminus \{z_1, \dots z_k\} \to \mathbb{C}$ analytic, γ contractible in A.

Then $$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f(z); z_j)\, I(\gamma; z_j).$$

proof: let $\sum_{n=0}^{\infty} a_n^{(j)} (z-z_j)^n + \sum_{m=1}^{\infty} b_m^{(j)} \frac{1}{(z-z_j)^m}$ be Laurent series $= S_1^{(j)}(z) + S_2^{(j)}(z)$ for $f(z)$ @ z_j.

$S_2^{(j)}(z)$ converges $\forall\, z \neq z_0$, uniformly for $|z - z_j| \geq \varepsilon$.

Thus $f(z) - \sum_{j=1}^{k} S_2^{(j)}(z)$ has removable singularities at each z_j

and Cauchy formula (Deformation Thm) applies:

$$\int_\gamma f(z) - \sum_{j=1}^{k} S_2^{(j)}(z)\, dz = 0$$

$$\Rightarrow \int_\gamma f(z)\,dz = \int_\gamma \sum_{j=1}^{k} S_2^{(j)}(z)\,dz$$

$$= \sum_{j=1}^{k} \int_\gamma S_2^{(j)}(z)\,dz = \sum_{j=1}^{k} \int_\gamma \sum_{m=1}^{\infty} b_m^{(j)} \frac{1}{(z-z_j)^m}$$

$$\sum_{j=1}^{k} \sum_{m=1}^{\infty} b_m^{(j)} \int_\gamma \frac{1}{(z-z_j)^m}\,dz$$

(by uniform conv. can interchange $\sum_m$ & $\int_\gamma$)

$$= \sum_{j=1}^{k} b_1^{(j)} \int_\gamma \frac{1}{z-z_j}\,dz$$

because all other powers of $z - z_j$ have antiderivatives

$$= \sum_{j=1}^{k} b_1^{(j)}\, 2\pi i\, I(\gamma; z_j)$$

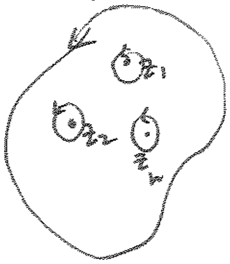
■

<u>version 2</u> let $\gamma = \partial\Omega$, Ω bd, p.w. C^1 bdry, $cl(\Omega) \subset A$, where $f : A \setminus \{z_1, \dots z_k\} \to \mathbb{C}$ is analytic then

$$\oint_{\partial\Omega} f(z)\,dz = 2\pi i \sum_{z_j \in \Omega} \operatorname{Res}(f; z_j).$$

this version ~~all~~ follows from the Green's Thm version of Cauchy's Thm, applied to $\Omega \setminus \bigcup_{z_j \in \Omega} D(z_j; \varepsilon_j)$ s.t. $D(z_j; \varepsilon_j) \subset \Omega$

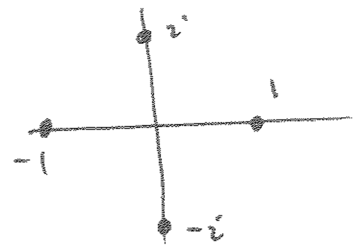
$\gamma = \partial\Omega$



$$\int_\gamma f(z)\,dz = \sum_{z_j \in \Omega} \oint_{|z-z_j| = \varepsilon_j} f(z)\,dz$$

use Laurent series @ z_j on each circle $|z - z_j| = \varepsilon_j$; integrate term by term, and result follows.

5a) $\displaystyle\oint_{|z|=2} \frac{5z^3}{z^4-1}$



$$z^4-1 = (z^2-1)(z^2+1)$$
$$= (z-1)(z+1)(z-i)(z+i)$$

so each pole is simple: $f(z) = \dfrac{g(z)}{h(z)}$; $\operatorname{Res}(f;z_0) = \dfrac{g(z_0)}{h'(z_0)}$

so integral $= 2\pi i\left(\cancel{5/4\,3}\ 4\cdot\frac{5}{4}\right)$

$\boxed{= 10\pi i}$

for us $g(z) = 5z^3$
$h(z) = z^4-1$
$\dfrac{g(z)}{h'(z)} = \dfrac{5z^3}{4z^3} = \dfrac{5}{4}$

5b) $\displaystyle\oint_{|z|=2} f(z)\,dz = -2\pi i \operatorname{Res}(f;\infty)$

$$= +2\pi i \operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right)$$

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left[\frac{\frac{5}{z^3}}{\frac{1}{z^4}-1}\right] = \frac{1}{z^2}\left[\frac{5z}{1-z^4}\right] \quad \left(\frac{z^4}{z^4}\right)$$

$$= \frac{5}{z}\left[\frac{1}{1-z^4}\right]$$

simple pole @ $z=0$ w residue $=5$.

so $2\pi i\left(\operatorname{Res}\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right)$

$= 2\pi i\cdot 5 = 10\pi i$

6a. $\displaystyle\int_\gamma \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \, dz$

$$= \sum_{m=1}^{\infty} b_m \int_\gamma \frac{1}{(z-z_0)^m} dz + \sum_{n=0}^{\infty} a_n \int_\gamma (z-z_0)^n dz$$

- we may interchange summation and integration because each series converges uniformly on γ.

except for $m=1$, all other $\displaystyle\int \frac{1}{(z-z_0)^m} dz = \frac{(z-z_0)^{-m-1}}{-m-1} + C$

& all $\displaystyle\int (z-z_0)^n dz = \frac{(z-z_0)^{n+1}}{n+1} + C$

have antiderivs, so integrals over closed contours yield zero.

and, $\displaystyle\int_\gamma \frac{1}{z-z_0} dz = 2\pi i$, as we know.

thus, the sum of integrals reduces to $b_1 \cdot 2\pi i$

6b). To pick off b_m with $m \neq 1$, multiply $f(z)$ by $(z-z_0)^{m-1}$ and integrate

$$\Rightarrow \int_\gamma f(z)(z-z_0)^{m-1} dz = 2\pi i\, b_m.$$

To pick off a_n, multiply by $\frac{1}{(z-z_0)^{n+1}}$ and integrate

$$\rightarrow \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz = 2\pi i\, a_n$$