

Math 4200
November 16, 2011
EXAM 2

Each complete problem below is worth 33 points. Choose any three out of the six problems to do. If you try more than three problems, indicate clearly which three you want graded. This exam is closed book and closed note, except for the residue table which I've provided. Show complete work for complete credit. Justify all steps in theorem proofs. Good Luck!

1) Consider

$$f(z) = \frac{2}{z} + \frac{3}{z-1}.$$

1a) Find the Laurent series for $f(z)$ in the punctured disk $0 < |z| < 1$.

(10 points)

1b) Find the Laurent series for $f(z)$ in the annulus $1 < |z| < \infty$.

(10 points)

1c) Use the Residue Theorem to compute

$$\int_{|z|=2} f(z) dz$$

where as usual the circle of radius 2 is traversed once counterclockwise. Explain how your answer is consistent with what you would get using the Laurent series in part b).

(13 points)

2a) State and prove Liouville's Theorem, for bounded entire functions. Hint: use the Cauchy Integral Formula for derivatives.

(15 points)

2b) Use Liouville's Theorem to prove the Fundamental Theorem of Algebra, which asserts that every degree n polynomial with complex coefficients factors completely over the complex number field, into n linear factors.

(18 points)

3) Compute the following contour integrals

3a)

$$\int_{|z|=8} \frac{\sin(2z)}{z^2} dz$$

(11 points)

3b)

$$\int_{|z|=8} z^2 \cdot e^{\frac{1}{z}} dz$$

(11 points)

3c)

$$\int_{|z|=8} \frac{z^2}{\sin(2 \cdot z)} dz$$

(11 points)

4) We proved two versions of the Residue Theorem. One version is for curves γ which are contractible in a domain A on which $f(z)$ is analytic except for at a finite number of singular points. The other version is for when $\gamma = \delta\Omega$ is the boundary of a bounded domain Ω compactly contained in A , and inside of which $f(z)$ has only a finite number of singularities. Both proofs use local Laurent expansions centered at the singular points, but in slightly different ways. State and prove either one of these two versions of the Residue Theorem.

(33 points)

5a) Let γ be the circle of radius 2, centered at the origin. Use the Residue Theorem to compute

$$\int_{\gamma} \frac{5z^3}{z^4 - 1} dz$$

(20 points)

(Hint: One of your formulas for residues at simple poles works especially well here!)

5b) Using residue at infinity and the domain exterior to γ , recompute the integral from part (a). Hint: Recall that if $f(z)$ is analytic outside of a sufficiently large disk, then

$$\text{Res}(f(z); \infty) := \text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right).$$

(13 points)

6) We proved that if $f(z)$ is analytic in an annulus $r < |z - z_0| < R$, then it has a Laurent series

$$f(z) = \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

And, conversely, we proved that any series of this type has a natural annulus of convergence so that each half of the Laurent series converges uniformly absolutely on compact subannuli

$$r < \rho \leq |z - z_0| \leq P < R.$$

6a) Let γ be a circle of radius greater than r and less than R , centered at z_0 and oriented counterclockwise. Prove that

$$\int_{\gamma} f(z) dz = 2\pi i b_1$$

(20 points)

(You may quote theorems which allow you to interchange a limit process with integration.)

6b) Extend your argument from 2a to show that every Laurent coefficient b_m, a_n is uniquely determined by the analytic function $f(z)$, by finding a formula for each such coefficient a_n or b_m . Hint: multiply $f(z)$ by appropriate powers of $(z - z_0)$.

(13 points)