Math 4200-1
Exam 1 sol'tns.

A) f is diff'ble @ z_0 iff $\lim\limits_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ exists; this limit is called $f'(z_0)$.

equivalent approx formula:

$$f(z_0+h) = f(z_0) + hf'(z_0) + he(h)$$

where $\lim\limits_{h\to 0} e(h) = 0$.

B) $$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

1 a) $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

CR: $u_x = v_y$
$u_y = -v_x$

f is analytic iff $F\begin{pmatrix}x\\y\end{pmatrix} := \begin{bmatrix}u(x,y)\\v(x,y)\end{bmatrix}$ is real diff'ble & CR hold

so, analytic $\Rightarrow$ CR
CR + F Real diff'ble $\Rightarrow$ analytic.

b) $u(x,y) = -x^4 + 6x^2y^2 - y^4$

$u_x = -4x^3 + 12xy^2$ $\qquad u_y = 12x^2y - 4y^3$
$u_{xx} = -12x^2 + 12y^2$ $\qquad u_{yy} = 12x^2 - 12y^2$

so $u_{xx} + u_{yy} = -12x^2 + 12y^2 + 12x^2 - 12y^2 = 0$
so u harmonic on $\mathbb{R}^2$

1c) $\left.\begin{array}{l} v_x = -u_y \\ v_y = u_x \end{array}\right\} \Rightarrow v_x = -12x^2y + 4y^3 \Rightarrow v = -4x^3y + 4y^3x + f(y)$

$v_y = u_x \Rightarrow \cancel{-4x^3} + \cancel{12y^2x} + f'(y) = \cancel{-4x^3} + \cancel{12xy^2}$
$\Rightarrow f'(y) = 0$
$\Rightarrow f(y) = C.$

1d). $u + iv = -x^4 + 6x^2y^2 - y^4 + i(-4x^3y + 4y^3x + C)$

in case $C = 0$ this is just $-(x+iy)^4 = -z^4!$

$= -\left[x^4 + 4x^3iy + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4\right]$
$= -\left[x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3)\right]$ ✓

2 a) $\log z = \ln|z| + i\arg z$

(need branch domain w/ branch pt @ $z=0$ to make this well-defined, since $\arg z$ is only defined up to multiples of 2π)

b) $e^{\log z} = e^{\ln|z| + i\arg z}$

$= e^{\ln|z|}\, e^{i\arg z}$

$= |z|\,(\cos\theta + i\sin\theta)$ where $\arg z = \theta$ (polar coords)

$= r\cos\theta + i\,(r\sin\theta)$ where $r = |z|$ (polar coords)

$= z$, expressed in polar coords

c) $e^{\log z} = z$, so $\log z$ is locally inverse fn of e^w, so in $\mathbb{C}$-diffble.

$\Rightarrow \frac{d}{dz} e^{\log z} = \frac{d}{dz} z = 1$

$\Rightarrow \underbrace{e^{\log z}}_{z} \frac{d}{dz}\log z = 1 \Rightarrow (\log z)' = \frac{1}{z}$.

d) use $\log w = \ln|w| + i\arg w$.

write $z = re^{i\theta}$ $0<\theta<2\pi$, so $\log z = \ln r + i\theta$

so $3\log z = 3\ln 3 + i(3\theta)$

then $z^3 = r^3 e^{3i\theta}$

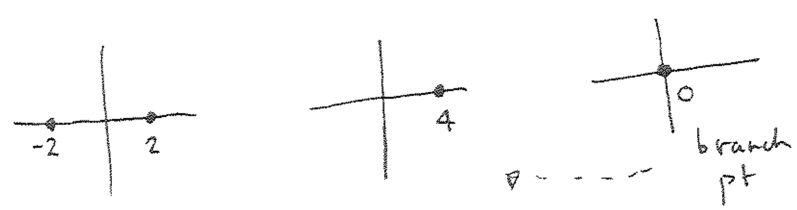
so $\log z^3 = \ln r^3 + i\arg(e^{3i\theta})$

$= 3\ln r + i\begin{cases} 3\theta & \text{if } 0<3\theta<2\pi \\ 3\theta - 2\pi & \text{if } 2\pi<3\theta<4\pi \\ 3\theta-4\pi & \text{if } 4\pi<3\theta<6\pi \end{cases}$

So we see that $\log z^3 = 3\log z$ only on the sector $0<3\theta<2\pi$, i.e. $0<\theta<\frac{2\pi}{3}$.

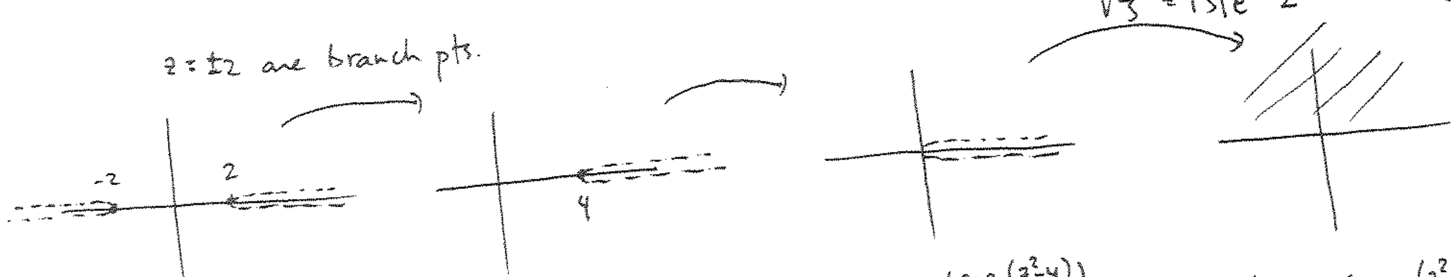
$f(z) = \sqrt{z^2-4}$

3 a) $z \longrightarrow z^2 = w \longrightarrow w - 4 = \zeta \longrightarrow \xi = \sqrt{\zeta}$



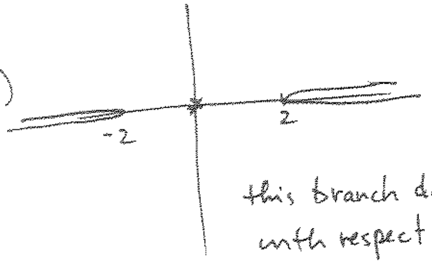
$z = \pm 2$ are branch pts.

$\sqrt{\zeta} = |\zeta|^{\frac{1}{2}} e^{\frac{i\arg\zeta}{2}}$ $0 < \arg$



so $\sqrt{z^2-4} := |z^2-4|^{\frac{1}{2}} e^{i\left(\frac{\arg(z^2-4)}{2}\right)}$ where $0<\arg(z^2$

3b)



this branch domain is star-shaped with respect to origin

(if $z = x+iy \in$ domain then the segment $F(s) = (1-s)(x+iy)$ is in the domain $0 \le s \le 1$;

$y \neq 0 \Rightarrow \mathrm{Im}(F(s)) \neq 0 \quad \forall\, 0 \le s < 1$ & F

$y = 0 \Rightarrow |x| < 2 \Rightarrow F(s) \in$ domain $\forall\, 0 \le s \le 1$.

thus domain is simply connected.

let $\gamma(t)$, $0 \le t \le 1$ a closed curve in domain. Then homotopy is

$$H(s,t) = (1-s)\gamma(t) \qquad 0 \le s \le 1$$

3c) Since domain is simply connected, the analytic fcn $\sqrt{z^2-4}$ has an antideriv

$$\Rightarrow \int_{|z|=1} \sqrt{z^2-4}\, dz = 0. \qquad \text{(Alternately use deformation thm.)}$$

4a) $$\int_\gamma z\,dz = \frac{1}{2}z^2\Big]_{-2}^{i} = -\frac{1}{2} - 2 = -5/2$$

b) let $\gamma(t) = (1-t)(-2) + ti \qquad 0 \le t \le 1$

$= 2(t-1) + ti$

$\gamma'(t) = 2 + i$

$$\int_\gamma z\,dz = \int_0^1 (2(t-1) + it)(2+i)\,dt = \int_0^1 \underbrace{4t-4-t}_{3t-4} + i\underbrace{(2t+2t-2)}_{4t-2}\,dt$$

$$= \frac{3}{2}t^2 - 4t\Big]_0^1 + \underbrace{i(2t^2-2t)\Big]_0^1}_{0}$$

$= 3/2 - 4$

$= -5/2$ ✓

c) let $\gamma : [a,b] \to A \subset \mathbb{C}$ be C^1, A open, $f : A \to \mathbb{C}$ cont.

if $\exists\, F : A \to \mathbb{C}$ s.t. $F' = f$, then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

pf $$\int_\gamma f(z)\,dz := \int_a^b \underbrace{f(\gamma(t))\gamma'(t)}_{\frac{d}{dt}F(\gamma(t))}\,dt$$ by $F' = f$ and chain rule for curves

$$= F(\gamma(t))\Big]_a^b = F(\gamma(b)) - F(\gamma(a)) \quad ∎$$

5). a) $\alpha(t)$ is a circle of radius 2, centered at i
and traversed once counterclockwise

b) $\gamma(t) = i + 2e^{it} + e^{4it}$
$\alpha(t) = i + 2e^{it}$

$$H(s,t) = i + 2e^{it} + (1-s)e^{4it} \qquad 0 \le s \le 1, \quad 0 \le t \le 2\pi.$$

$$H(s,t) - i = 2e^{it} + (1-s)e^{4it}$$

$$\Rightarrow |H(s,t) - i| \geq |2e^{it}| - |1-s||e^{4it}| \qquad \text{reverse } \Delta \text{ ineq.}$$
$$= 2 - |1-s|$$
$$\geq 2-1$$
$$= 1 \qquad \Rightarrow |H(s,t)-i| \geq 1.$$

c) $$\int_\alpha \frac{1}{z-i}\,dz = \int_0^{2\pi} \frac{1}{\alpha(t)-i}(\alpha'(t))\,dt$$
$$= \int_0^{2\pi} \frac{1}{2\cancel{e^{it}}}\, 2i\cancel{e^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

d) Let $f(z)$ be analytic on the (open) domain A.
Let γ_0, γ_1 be homotopic as closed curves in A (& let γ_0, γ_1 be piecewise C^1).
Then $$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

in our case, $A = \mathbb{C} \setminus \{i\}$, $f(z) = \frac{1}{z-i}$, $\gamma_0 = \alpha$, $\gamma_1 = \gamma$

So $$\int_\gamma \frac{1}{z}\,dz = \int_\alpha \frac{1}{z}\,dz = 2\pi i$$

6a) 

C^1
let $f(z)$ analytic on an open domain containing the bd domain A, with p.w. C^1 boundary curves.

since $$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy$$

since u,v are C^1 we may apply Green's to ∂A: (properly orien

$$\int_{\partial A} f(z)\,dz = 0 \Leftarrow \begin{cases} \int_{\partial A} u\,dx - v\,dy = \iint_A (-v)_x - u_y = \iint_A \underbrace{-v_x - u_y}_{0 \text{ since } u_y = -v_x \text{ CR}}\,dA = 0 \\ \int_{\partial A} v\,dx + u\,dy = \iint_A \underbrace{u_x - v_y}_{=0 \text{ since } u_x = v_y \text{ CR}}\,dA = 0. \end{cases}$$

(6b). easiest way. $\frac{1}{z(z-1)}$ is analytic for $|z|>1$

and $|z|=2$ is homotopic as closed curve in this domain to $|z|=R$ $R\geq 2$

$$\Rightarrow \oint_{|z|=2} \frac{1}{z(z-1)}\,dz = \underbrace{\oint_{|z|=R} \frac{1}{z(z-1)}\,dz}$$

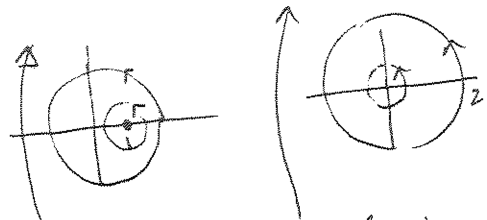
$$|\quad| \leq \oint_{|z|=R} \frac{1}{|z||z-1|}\,|dz| \leq \oint_{|z|=R} \frac{1}{R(R-1)}\,|dz| = \frac{2\pi \cancel{R}}{\cancel{R}(R-1)} \xrightarrow{\text{as } R\to\infty} 0$$

thus original integral $=0$.

or use part. frac.

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$$

So $$\int_{|z|=2} \frac{1}{z(z-1)}\,dz = \int_{|z|=2} \frac{1}{z-1}\,dz - \int_{|z|=2} \frac{1}{z}\,dz$$



$$= \oint_{|z-1|=1/2} \frac{1}{z-1}\,dz = 2\pi i \qquad = -\int_{|z|=1} \frac{1}{z}\,dz = -2\pi i$$

So sum is zero.