EXAM 1

Math 4200−1   October 5, 2011

Answer problems (A) and (B) below, which are each worth 5 points. Then choose any 3 of the 6 multi−part problems which follow, each of which is worth a total 30 points. If you try more than 3 of these problems, indicate clearly which 3 you want graded.This exam is closed book and closed note. Show complete work for complete credit. Good Luck!

Mandatory problems:

A) Define what it means for a function $f(z)$ to be complex differentiable at a point $z_0 \in \boldsymbol{C}$, and also write down the affine approximation formula (with error estimate) which is equivalent to this definition.

(5 points)

B) Let $\gamma : [a, b] \to \boldsymbol{C}$ be a C^1 curve, and let $f : A \to \boldsymbol{C}$ be a continuous function from an open domain $A \subseteq \boldsymbol{C}$ containing the image of γ. Define

$$\int_\gamma f(z)dz .$$

(5 points)

Choose 3 out of 6 problems:

1a) What are the Cauchy−Riemann equations? Explain precisely how they are related to complex differentiability (proofs not required).

(8 points)

1b) Show that the function $u(x, y) = -x^4 + 6 \cdot x^2 \cdot y^2 - y^4$ is harmonic.

(5 points)

1c) Find the harmonic conjugate $v(x, y)$

(10 points)

1d) Identify the analytic function $f(x + i \cdot y) = u(x, y) + i \cdot v(x, y)$ as an elementary complex analytic function that you recognize.

(7 points)

2a) Define $\log(z)$.

(5 points)

2b) Prove $e^{\log(z)} = z$, no matter what branch of the argument is used for the logarithm.

(8 points)

2c) Use (2b) and the chain rule to deduce that the derivative of $\log(z)$ is $\dfrac{1}{z}$.

(7 points)

2d) If you choose branch of the logarithm, so that $0 < arg(z) < 2 \cdot \pi$, then explain for which z the following identity does hold, and for which z it doesn't:

$$\log z^3 = 3 \log z$$

(10 points)

Hint: Write $z = r \cdot e^{i\theta}$ with $0 < \theta < 2\pi$, and notice that the two sides can differ at most by integer multiples of $2 \cdot \pi \cdot i$.

3a) Define a branch of the function

$$f(z) = \sqrt{z^2 - 4}$$

on a simply connected branch domain consisting of the complex plane complement two rays, and containing the closed unit disk $|z| \leq 1$.

(10 points)

3b) Explain why your branch domain is simply connected, by exhibiting a general homotopy formula which will homotopy all closed curves to a point, through closed curves.

(10 points)

3c) For this branch of $\sqrt{z^2 - 4}$, what is

$$\int_{|z| = 1} \sqrt{z^2 - 4}\, dz\, ,$$

where the unit circle is traversed once counterclockwise? Justify your answer using Theorems we've proved in this class.

(10 points)

4a) Let γ be any piecewise C^1 curve from the point -2 to the point i in the complex plane. Using the fundamental theorem of Calculus for contour integration, deduce the value of

$$\int_{\gamma} z\, dz$$

(10 points)

4b) Explicitly compute the contour integral above by using any piecewise C^1 (or C^1) curve you like which connects -2 to i .

(10 points)

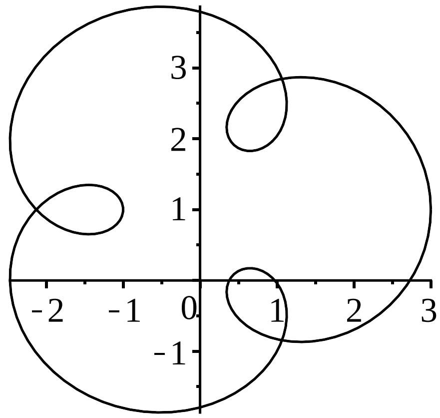
4c) State and prove the fundamental theorem of Calculus, for contour integrals of analytic functions with antiderivatives, for C^1 contours.

(10 points)

5) Let

$$\gamma(t) = i + 2\,e^{it} + e^{4it}, \; 0 \le t \le 2\cdot\pi .$$

In case you're having trouble visualizing the image curve, here is a picture of it:



5a) Carefully sketch the (image of the) curve $\alpha(t) = i + 2\cdot e^{it}, \; 0 \le t \le 2\cdot\pi$ onto the sketch above, making it clear with words and your sketch what geometric object is traced out by the curve α (include this question sheet with the solutions you hand in).

(5 points)

5b) Find a homotopy through closed curves, from $\gamma(t)$ to the curve $\alpha(t)$ above, such that the homotopy avoids the point i. Make estimates to prove that your homotopy does avoid i.

(10 points).

5c) Using the definition of contour integral (problem B above), compute

$$\int_{\alpha} \frac{1}{z-i}\, dz$$

(10 points)

5d) State and use the deformation theorem for closed curves, to deduce the value of

$$\int_{\gamma} \frac{1}{z-i}\, dz$$

(10 points)

6a) Use Green's Theorem to prove that the sum of the contour integrals (properly oriented) around the piecewise C^1 boundary curves of a bounded domain A (possibly containing holes) is zero, assuming the integrand function $f(z)$ is analytic and C^1 on a larger domain containing the closure of A.

(20 points)

6b) Using the theorem in (6a), or any other method you can justify, deduce the value of

$$\int_{|z| = 2} \frac{1}{z\cdot(z-1)}\, dz$$

(10 points)