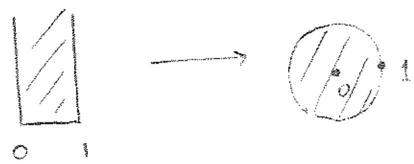
Math 4200
Fri 12/2

Let's make the §5.1-5.2 HW due next Friday, so our usual problem session times are helpful next week

Continue conformal transformations...

Do all the steps carefully, for compositions leading to



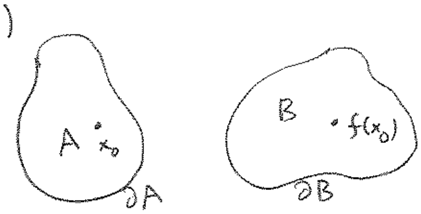
Magic Theorem (I think the text garbles the statement on page 322).

in $\mathbb{R}^n$: Let $A, B \subset \mathbb{R}^n$, each open, connected, bounded.

Let $f: A \to \mathbb{R}^n$, $f \in C^1$ with $df_x : T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^n$ invertible $\forall x \in A$

- $f: \bar{A} \to \mathbb{R}^n$ continuous and 1-1
- $f(\partial A) = \partial B$
- $f(x_0) \in B$ for some $x_0 \in A$.

Then $f(A) = B$ and f is a global diffeomorphism between A and B (i.e. $\exists f^{-1}: B \to A$ differentiable).
(and homeomorphism for $\bar{A}$ to $\bar{B}$)
i.e. f^{-1} is continuous



proof: Step 1: $f(A) \subset B$

proof: Let $\mathcal{O} = \{x \in A \text{ s.t. } f(x) \in B\}$
- $x_0 \in \mathcal{O}$
- $\mathcal{O}$ is open by local inverse function theorem, since $x_1 \in \mathcal{O}$, $f(x_1) \in B \Rightarrow f^{-1}: B_\varepsilon(f(x_1)) \to N_{x_1}$ $\exists$.
- $\mathcal{O}$ is closed in A because if $\{x_k\} \subset \mathcal{O}$, $x_k \to x \in A$
then $\underset{B}{\underset{\cap}{f(x_k)}} \to f(x) \in \bar{B}$. If $f(x) \in \partial B$ then also $f(\tilde{x}) = f(x)$, $\tilde{x} \in \partial A$ violates f 1-1. Thus $f(x) \in B$, i.e. $x \in \mathcal{O}$.

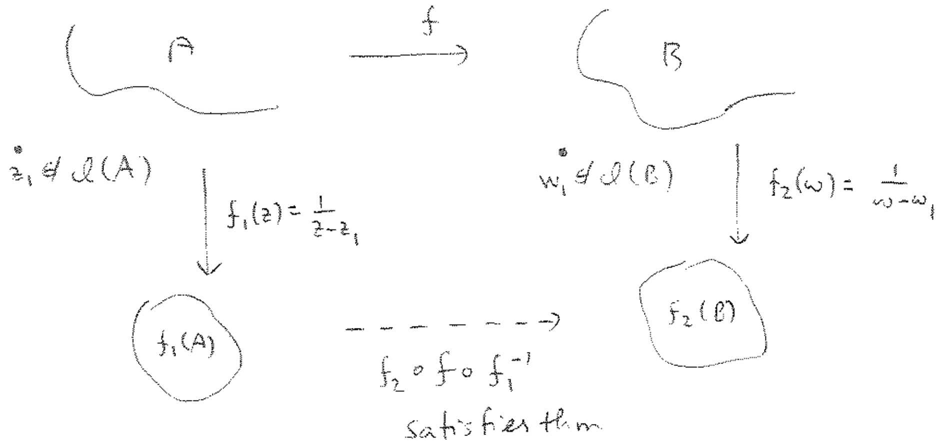
Step 2: $f(A) = B$

proof $f(A)$ is open (because of local inverse function again)

So $f(A) \subset B$ is open
But $f(A)$ is also closed in B because if $\{f(x_k)\} = \{y_k\} \subset f(A)$, with $\{y_k\} \to y \in B$
then subseq. $\{x_{k_j}\} \to x \in \bar{A}$, with $f(x) = y$.
$x \notin \partial A$ because $f(\partial A) = \partial B$.
thus $x \in A$ and $y \in f(A)$.

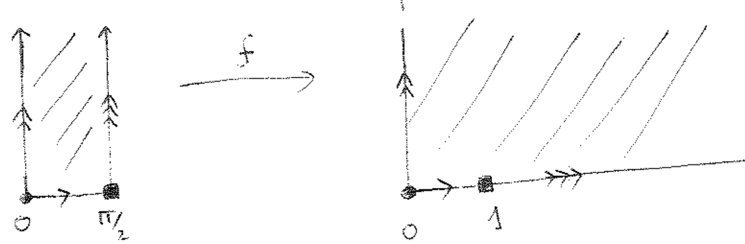
Thus $f(A) = B$

Remark: In $\mathbb{C}$ you can also apply this theorem to unbounded domains, (i.e. in $\mathbb{C} \cup \{\infty\}$), because of the following diagram



Example: (short-cut for the example we started with)

Show $f(z) = \sin z$ is a conformal equivalence



- check boundary: use $\sin z = \sin(x+iy) = \sin x (\cos iy) + \cos x \sin(iy)$
$= \sin x \cosh y + i \cos x \sinh y$
(arrows + if orientation preserving also shows $\exists\, z_0 \in A$, $f(z_0) \in B$)

- 1-1: if $\sin z_1 = \sin z_2$
$\sin x_1 \cosh y_1 + i \cos x_1 \sinh y_1 = \sin x_2 \cosh y_2 + i \cos x_2 \sinh y_2$
may assume $x_1 \le x_2$. $\Rightarrow \sin x_1 \le \sin x_2$
compare real parts $\Rightarrow \cosh y_1 \ge \cosh y_2 \Rightarrow y_1 \ge y_2$
compare imag parts: $\cos x_1 \ge \cos x_2$, $\Rightarrow \sinh y_1 \le \sinh y_2 \Rightarrow y_1 \le y_2$
thus $y_1 = y_2$. thus $x_1 = x_2$

- $\frac{d}{dz} \sin z = \cos z \neq 0$ in A.

(If there's time today, else next week?)

It's time to consider $S^2_{\mathbb{R}}$ as a <u>complex manifold</u>: (a.k.a. "Riemann surface" "complex curve")

Our atlas has 2 charts, the z-plane $\mathbb{C}_z$
& the w-plane $\mathbb{C}_w$

$$S^2_{\mathbb{R}} := \frac{\mathbb{C}_z \cup \mathbb{C}_w}{\sim}$$

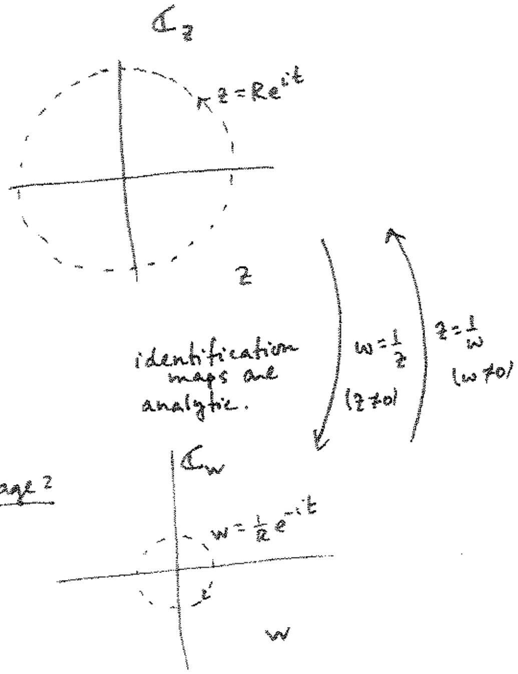
where the equivalence relation is that

$$z \sim \frac{1}{w} \quad \text{for } z, w \neq 0.$$

$z = \infty$ now means $w = 0$
($w = \infty$ means $z = 0$)

example: $z = 2$ is the same point as $w = \frac{1}{2}$

<u>Atlas page 1</u>



<u>Atlas page 2</u>

(Think of charts of S^2 obtained by stereographic proj. from north & south poles; it's almost the same.)

Def: $f : \underset{A}{\overset{S^2_R}{\vee}} \longrightarrow S^2_R$ is analytic at $p_0 \in A$ iff for at least one (hence all) of the 2–4 possible representative maps from chart to chart is analytic at the corresponding z_0 or w_0:

inclusion map

$$\begin{tikzcd}
 & S^2_R \arrow[rrr, "f"] & & & S^2_R & \\
\mathbb{C}_z \arrow[ur, "i_z"] \arrow[r, bend left, "w=\frac{1}{z}"] & \mathbb{C}_w \arrow[u, "i_w"'] \arrow[l, bend left, "z=\frac{1}{w}"] & & \mathbb{C}_z \arrow[ur, "i_z"] \arrow[r, bend left, "w=\frac{1}{z}"] & \mathbb{C}_w \arrow[u, "i_w"'] \arrow[l, bend left, "z=\frac{1}{w}"] &
\end{tikzcd}$$

$i_z^{-1} \circ f \circ i_z : \mathbb{C}_z \to \mathbb{C}_z$

$i_z^{-1} \circ f \circ i_w : \mathbb{C}_w \to \mathbb{C}_z$

$i_w^{-1} \circ f \circ i_z : \mathbb{C}_z \to \mathbb{C}_w$

$i_w^{-1} \circ f \circ i_w : \mathbb{C}_w \to \mathbb{C}_w$

Def: $f : \underset{A}{\overset{S^2_R}{\vee}} \to S^2_R$ is analytic on A if f is analytic at each $p \in A$.

Example Let f have a pole at $z_0 \in \mathbb{C}_z$.

Then considered as a map to the Riemann sphere, f is analytic at z_0!

pf since $f(z_0) = \infty$ we consider

$$(i_w \circ f \circ i_z)(z) = \frac{1}{f(z)} \quad : \mathbb{C}_z \to \mathbb{C}_w.$$

$$f(z) = \sum_{m=1}^{N} \frac{b_m}{(z-z_0)^m} + \sum_{n=0}^{\infty} a_n (z-z_0)^n = \frac{1}{(z-z_0)^N} \phi(z) \qquad \begin{array}{l} \phi(z_0) = b_N \neq 0 \\ \phi \text{ analytic near } z_0 \end{array}$$

$$\Rightarrow \frac{1}{f(z)} = (z-z_0)^N \frac{1}{\phi(z)}$$ has a zero of order N (and is analytic in a nbhd of z_0)!

$\blacksquare$

Example: $f(z)$ is analytic @ $z = \infty$ iff $\begin{cases} f(\frac{1}{w}) & \text{is analytic @ } w=0, \text{ in case } f(\infty) \neq \infty \\ \frac{1}{f(\frac{1}{w})} & \text{is analytic @ } w=0, \text{ in case } f(\infty) = \infty \end{cases}$

Corollary (after you check all cases), Let $f(z) = \dfrac{p_n(z)}{q_m(z)}$ be a rational function. Then $f(z)$ extends to S^2_R, as an analytic map $: S^2_R \to S^2_R$.

[In fact, one can show these are the only analytic functions from $S^2_R \to S^2_R$! And, they are $\max(m,n)$ to 1, so are bijections iff $m=n=1$ FLT's!]

amazing complex analysis / geometry / group theory (algebra) interplay in the study of FLT's.

$$f(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{C}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1 \quad \text{(after normalizing)}$$

$f(z)$ is a conformal diffeomorphism of S^2_R, and all (orientation preserving) conformal diffeos of S^2_R are given by such an f

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ s.t. } a,b,c,d \in \mathbb{C},\ |A| = 1 \right\} := SL(2,\mathbb{C})$$

special linear group of 2×2 matrices with entries in $\mathbb{C}$.

write $[f] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. for the matrix associated to FLT f.

Interesting <u>subgroups</u> of the conformal diffeos... (and corresponding matrices)

- $f(z) = re^{i\theta}z + b$ $\qquad \begin{bmatrix} \sqrt{r}\,e^{i\theta/2} & \sqrt{r}\,e^{i\theta/2} b \\ 0 & \frac{1}{\sqrt{r}} e^{-i\theta/2} \end{bmatrix}$

  $r > 0,\ b \in \mathbb{C}$

  rotations composed with translations

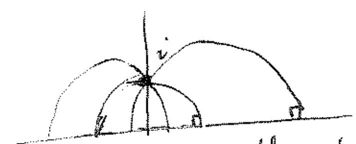
  $\mathbb{C}$: these are the only orientation pres. conformal diffeos from $\mathbb{C} \to \mathbb{C}$ (HW).

  Geom: these are the only orientation preserving <u>isometries</u>: $\mathbb{C} \to \mathbb{C}$ $(\mathbb{R}^2 \to \mathbb{R}^2)$ where we measure arclength by $ds = |dz|$

- $f(z) = \dfrac{az+b}{cz+d}$ $\qquad a,b,c,d \in \mathbb{R},\ |A| = 1$ $\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  $\mathbb{C}$: all conformal diffeos of upper half plane

  Geom: isometries of the hyperbolic plane, $\{x+iy \text{ s.t. } y>0\}$ $\qquad |ds| = \frac{1}{y}$



geodesics thru i

- $f(z) = \dfrac{z - z_0}{1 - \bar{z}_0 z}$ $\qquad z_0 \in D(0;1)$

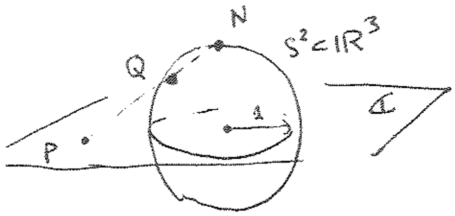
  $\mathbb{C}$: all conformal diffeos of $D(0;1)$ (o.p.)

  Geom: isometries (o.p.) of the hyperbolic disk (which is isometric to hyperbolic plane, e.g. via $f(z) = \frac{z-i}{z+i}$)

Notice we didn't talk about the geometry associated to the full group of FLT's. Since they <u>are</u> the o.p. conformal diffeo group of S^2_R, the geometry must have to do with the round sphere...

standard identification of $S^2_{\mathbb{R}}$ with $S^2 = \{x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$ via (inverse) stereographic projection

(z = x_3^2)



St : stereographic projection from $\begin{pmatrix}0\\0\\1\end{pmatrix} = N$ $(= \infty)$

$St(Q) = P$

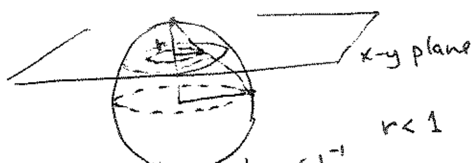
$St^{-1}(P) = Q$

one can check that St & St^{-1} are conformal and that $\{\text{circles}, \text{lines}\} \subset \mathbb{C}$ correspond to $\{\text{circles}\} \subset S^2$

the only Euclidean isometries of S^2 are rotations (orient. preserving).
So how can the geometry of S^2 have anything to do with FLT's?

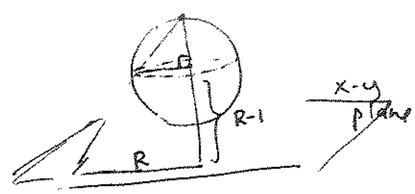
Here's how: You can stereographically project from the north pole of <u>any</u> (unit) sphere in $\mathbb{R}^3$, back to the x-y plane.
We checked dilations, rotations, translations, inversions generate all FLT's. They all <u>have</u> to do with St and isometries of $S^2 \subset \mathbb{R}^3$!

dilations : lift or lower S^2



$St \circ \text{lower} \circ St^{-1}$

$T_1(z) = r x$



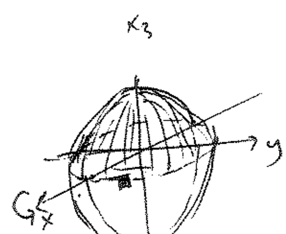
$St \circ \text{lift} \circ St^{-1}$

$T_1(z) = Rz$ $R > 1$

rotations in $\mathbb{C}$: $St \circ \overset{\text{rotate}}{S^2} \circ St^{-1}$

translations in $\mathbb{C}$: $St \circ \overset{\text{horiz}}{\text{trans}} \circ St^{-1}$

inversions : St^{-1}, then rotate π about the x-axis, then St. (Easy to check this works on unit circle!



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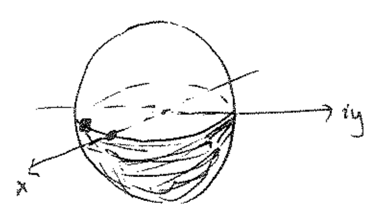

Can you see what a map from the unit disk to the upper half space would correspond to?

[I couldn't find good on-line visualizations of this, not even at wikipedia] [I'll be happy if anyone can find a link!].