

Math 4200  
Wed Aug 31

§1.4 cont'd: adding functions to the mix of sets, sequences.

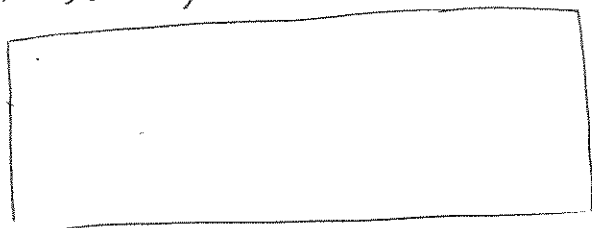
We'll be interested in

$$\begin{array}{ccc}
 f: A \rightarrow \mathbb{C} & , & \gamma: I \rightarrow \mathbb{C} & & w: A \rightarrow \mathbb{R} \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{C} & & \mathbb{R} & & \mathbb{C} \\
 \text{complex fns} & & \text{curves} & & \text{real valued functions}
 \end{array}$$

Since continuity is easily discussed for  $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we will temporarily adopt that generality, and use  $B(x_0; \epsilon) := \{x \in \mathbb{R}^n \text{ s.t. } \|x - x_0\| < \epsilon\}$  notation (rather than disk notation  $D(z_0; \epsilon)$ )

Def  $F: A \rightarrow \mathbb{R}^m$ ; For  $B \subset A$   $f(B) := \{y \in \mathbb{R}^m \text{ s.t. } y = f(x) \text{ for some } x \in B\}$   
For  $B \subset \mathbb{R}^m$ ,  $f^{-1}(B) := \{x \in A \text{ s.t. } f(x) \in B\}$

Def  $F: A \rightarrow \mathbb{R}^m$ . F is continuous at  $x_0$  iff



Def F is continuous on A iff F is cont. at  $x_0, \forall x_0 \in A$

Def  $U \subset A$  is relatively open (w.r.t. A) iff  $U = V \cap A, \forall V \subset \mathbb{R}^n$  open  
 $W \subset A$  is relatively closed iff  $W = V \cap A, \forall V$  closed.

Theorem: The following are equivalent, for  $x_0 \in A, F: A \rightarrow \mathbb{R}^m$

- (1) F is continuous at  $x_0$
  - (2) F is sequentially continuous at  $x_0$ , i.e.  $\forall \{x_n\} \subset A, \overset{\text{such that}}{\{x_n\}} \rightarrow x_0$  then  $\{f(x_n)\} \rightarrow f(x_0)$
- (see thm. §1.4 #18)

Theorem: The following are equivalent, for  $F: A \rightarrow \mathbb{R}^m$

- (1) F is continuous on A ( $\epsilon$ - $\delta$  def.)
- (2)  $F^{-1}(\Theta)$  is (relatively) open in A,  $\forall \Theta$  open in  $\mathbb{R}^m$

this is an important and useful characterization of continuous.  
there's room on next page for proof.

Theorem: If the range of  $F, G$  is  $\mathbb{C}$  ( $\cong \mathbb{R}^2$ ), and  $F, G$  is written in complex form

and if  $F, G$  have common domain  $A$ .  
Then  $F, G$  continuous at  $x_0$  implies  $x_0 \in A$ .

- (1)  $F+G$  cont at  $x_0$
- (2)  $FG$  cont at  $x_0$
- (3)  $F/G$  cont at  $x_0$  if  $G(x_0) \neq 0$

Continuous functions and compactness:

(1) Thm: If  $K$  is compact,  $F: K \rightarrow \mathbb{R}^m$  continuous  
 $\bigcap \mathbb{R}^n$  Then  $F(K)$  is compact  $\bigcap \mathbb{R}^m$

(2) Corollary: If  $K$  is compact,  $F: K \rightarrow \mathbb{R}^1$   
 $\bigcap \mathbb{R}^n$  then  $F$  attains its minimum (infimum) "m"  
& maximum (supremum) "M"  
i.e.  $\exists x_1 \in K$  s.t.  $F(x_1) = \inf_{x \in K} F(x)$   
 $\exists x_2 \in K$  s.t.  $F(x_2) = \sup_{x \in K} F(x)$

Def:  $F: \overset{\mathbb{R}^n}{A} \rightarrow \mathbb{R}^m$  is uniformly continuous iff

(3)

(3) Theorem:  $K \overset{\mathbb{R}^n}{\text{compact}}$ ,  $f: K \rightarrow \mathbb{R}^m$   
continuous  
 $\Rightarrow f$  uniformly continuous

### Connectivity and functions

(1) Thm.  $A \overset{\mathbb{R}^n}{\text{connected}}$ ,  $F: A \rightarrow \mathbb{R}^m$  continuous  $\Rightarrow F(A)$  connected

Def  $\overset{\mathbb{R}^n}{A}$  is path connected iff  $\forall x, y \in A \exists \gamma: [a, b] \rightarrow A$  continuous  
(a "path")  
s.t.  $\gamma(a) = x, \gamma(b) = y$

(2) Thm  
 $A$  path connected  $\Rightarrow A$  connected

(3) Thm  $A$  open, connected  $\Rightarrow A$  path connected

final theorem today (really about compactness and functions)

Thm (positive distance theorem).

Let  $K$  compact,  $\mathcal{O}$  open,  $K \subset \mathcal{O}$

Then  $\exists \delta > 0$  s.t.

$$\forall x \in K, B(x; \delta) \subset \mathcal{O}.$$

