Math 4200
Fri 26 Aug

§1.3 "Basic" complex functions

HW for Fri 2 Sept:
§1.3 1a, 4b, 5a, 6a, 7a, 9, 10, 16b, 21, 23, 30b
§1.4 1, 2b, 3, 4, 5, 8, 11, 13, 16, 18
(§1.4 is a review of most of the 3220 material we'll need in this course)

<u>Example 1</u> $f(z) = az + b$

- polar form

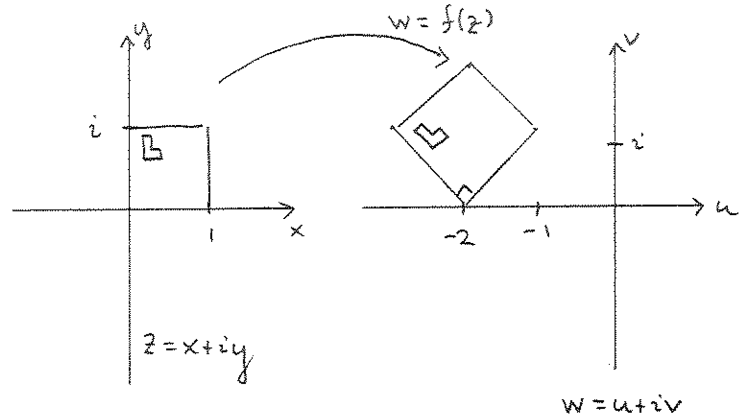
$z = |z|e^{i\theta}$ $\theta = \arg(z)$
$a = |a|e^{i\phi}$ $\phi = \arg(a)$

$$f(z) = |a|e^{i\phi}|z|e^{i\theta} + b$$
$$= |a||z|e^{i(\theta+\phi)} + b$$

f is a composition of
- scaling by a factor of $|a|$

then, • rotating by an angle ϕ
then, • translating by b.

e.g. $f(z) = (1+i)z - 2$
$= \sqrt{2}\, e^{i\pi/4} z - 2$



The transformation $\Phi : \mathbb{C} \to \mathbb{R}^2$ is a real vector space isomorphism,
$\Phi(z) = \begin{bmatrix} \mathrm{Re}\, z \\ \mathrm{Im}\, z \end{bmatrix}$ with inverse $\Phi^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + iy$

Each $f : \mathbb{C} \to \mathbb{C}$ corresponds to $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $F\left(\begin{bmatrix} \mathrm{Re}\, z \\ \mathrm{Im}\, z \end{bmatrix}\right) := \begin{bmatrix} \mathrm{Re}(f(z)) \\ \mathrm{Im}(f(z)) \end{bmatrix}$

i.e. if $z = x+iy$ and $f(z) = w = u+iv = u(z) + iv(z)$
then $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$

<u>Example 2</u>: Re-analyze Example 1 in terms of the corresponding $F : \mathbb{R}^2 \to \mathbb{R}^2$

Begin with
$z = x + iy$
$a = \alpha_1 + i\alpha_2$
$b = \beta_1 + i\beta_2$

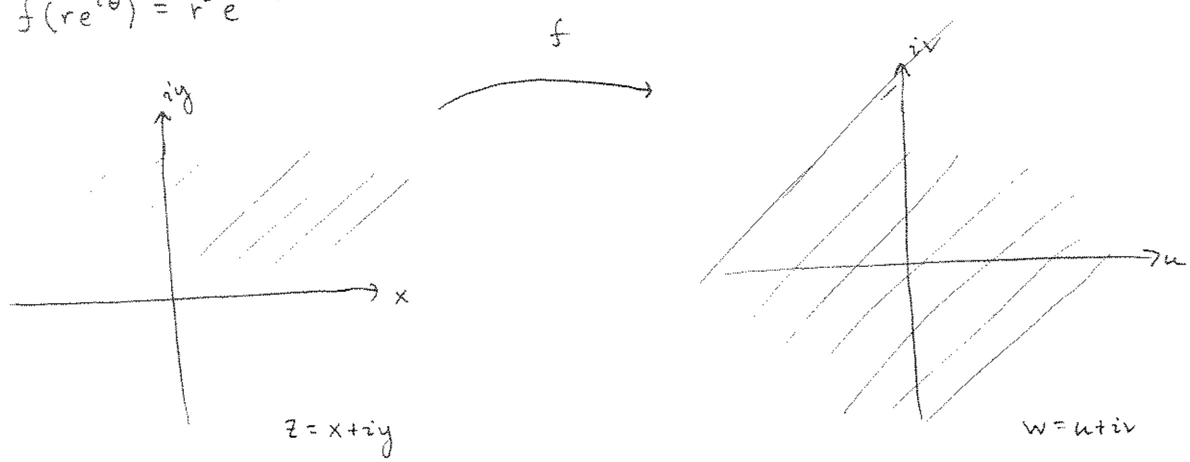
$$f(x+iy) = (\alpha_1 + i\alpha_2)(x+iy) + (b_1 + ib_2)$$

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$$

This should look familiar to you!

Example 3 $f(z) = z^2$

$f(re^{i\theta}) = r^2 e^{2i\theta}$



$z^2 = w$

$z = \pm\sqrt{w}$

↑

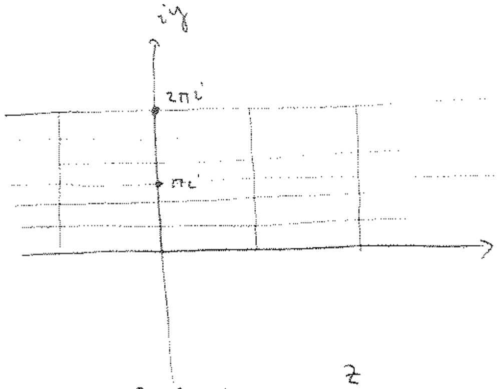
2 different choices possible.

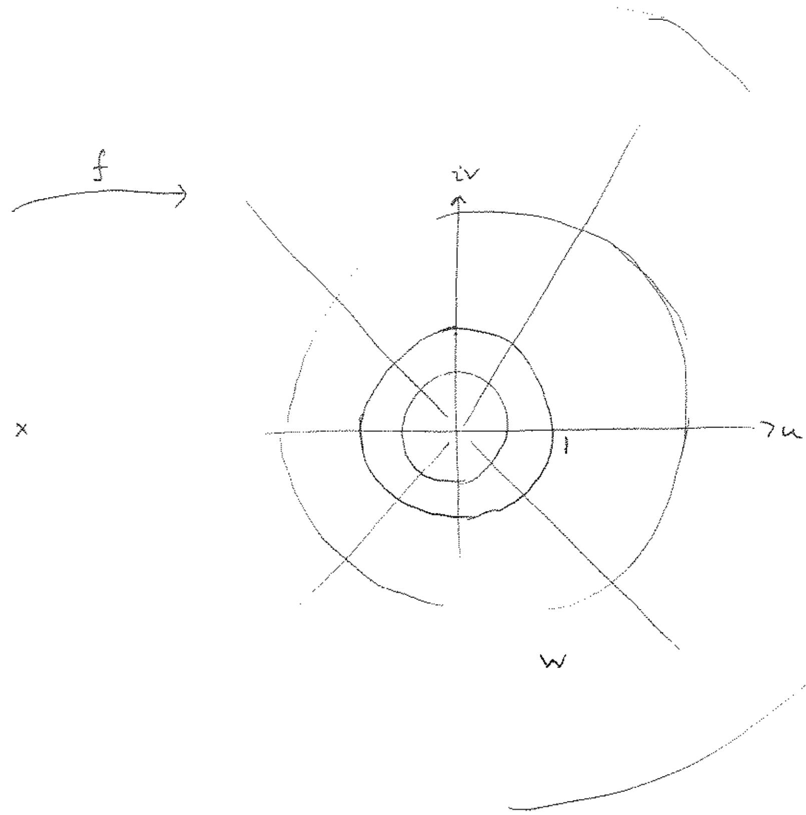
Example 4 $f(z) = z^3$

$f(re^{i\theta}) = r^3 e^{i(3\theta)}$

Example 5 $z = x+iy$

$$\boxed{f(z) = e^z := e^x e^{iy}}$$





property: $e^{z+w} = e^z e^w$
did we check this?

inverse fcn?

$$e^z = w$$

$$e^x(\cos y + i \sin y) = u + iv$$

$$\Longleftrightarrow \begin{cases} |w| = e^x \Rightarrow x = \log|w| \quad (\ln|w|) \\ y = \arg(w) \end{cases}$$

so (switching letters),

Example 6 $\boxed{\log z := \log|z| + i \arg z}$

if, for example, we restrict $0 \le \arg z < 2\pi$
then we get a "branch" of the log function which
maps $\mathbb{C} \setminus \{0\}$ 1-1 and onto the (half open) strip above

and $\log(e^z) = z$, $e^{\log z} = z$ with these restrictions

Many other "branch" choices exist!

verify!

also, log property:

$$\log zw = \log z + \log w$$

(up to a multiple of 2π in imag. part)

power functions

For a given branch of log, define

$$z^w := e^{w \log z} \qquad \text{(uniquely defines } z^w \text{ for the given branch of log)}$$

In general z^w is multiple-valued, because $\log z$ is

- If n is integer, $z^n := e^{n \log z} = e^{n(\log|z| + i\overbrace{\arg z}^{\pm 2\pi k})}$

  using the power definition above

  $= \left[e^{\log|z| + i(\arg z + 2\pi k)}\right]^n \qquad (e^{z+w} = e^z e^w)$

  $= z^n$ since $e^{2\pi i k} = \cos 2\pi k + i \sin 2\pi k$
  $= 1$ for k an integer

  i.e. $\underbrace{(z)(z)\cdots(z)}_{n}$

- If $w = \frac{p}{q}$ is a rational in lowest terms, (p, q no common divisors)

  then $z^{p/q}$ is one of the possible q values of (previous) p/q^{th} roots, i.e.

  $$z = re^{i\theta}$$
  $$z^{p/q} = r^{p/q} e^{ip(\cdot \frac{1}{q}(\theta + 2\pi k))} \qquad k = 0, 1, \dots q-1$$

  you can check this on your own!

- If w is irrational,

  z^w has ∞'ly many possible values, depending on log branch

  (but is uniquely defined after you specify a branch of logarithm)

Trig fcns

If x is real, $e^{ix} = \cos x + i\sin x$ Eqtn 1

$e^{-ix} = \cos x - i\sin x$ Eqtn 2

Also

$\dfrac{\text{Eqtn 1} + \text{Eqtn 2}}{2} \Rightarrow \cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$; $\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$

$\dfrac{\text{Eqtn 1} - \text{Eqtn 2}}{2i} \Rightarrow \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$; $\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$

$\cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ $\qquad$ $\cosh z := \frac{1}{2}\left(e^{z} + e^{-z}\right) = \cos(iz)$

$\sin z := \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ $\qquad$ $\sinh z := \frac{1}{2}\left(e^{z} - e^{-z}\right) = \frac{1}{i}\sin(iz)$

or

$$\left(\begin{array}{l} \cos z = \cosh(iz) \\ \sin z = \frac{1}{i}\sinh(iz) \end{array}\right).$$

Trig identities hold! (So also trigh identities)

$\sin^2 z + \cos^2 z = 1$

$\sin(z+w) = \sin z\cos w + \cos z\sin w$

$\cos(z+w) = \cos z\cos w - \sin z\sin w$

trigh...

$\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1 \ldots$