Math 4200-1
Wed 8/24
§1.2
Thursday problem session (tomorrow)
11:30 - 12:45 JTB 320
for HW due Friday.

Monday we showed that for

$$e^{i\theta} := \cos\theta + i\sin\theta$$

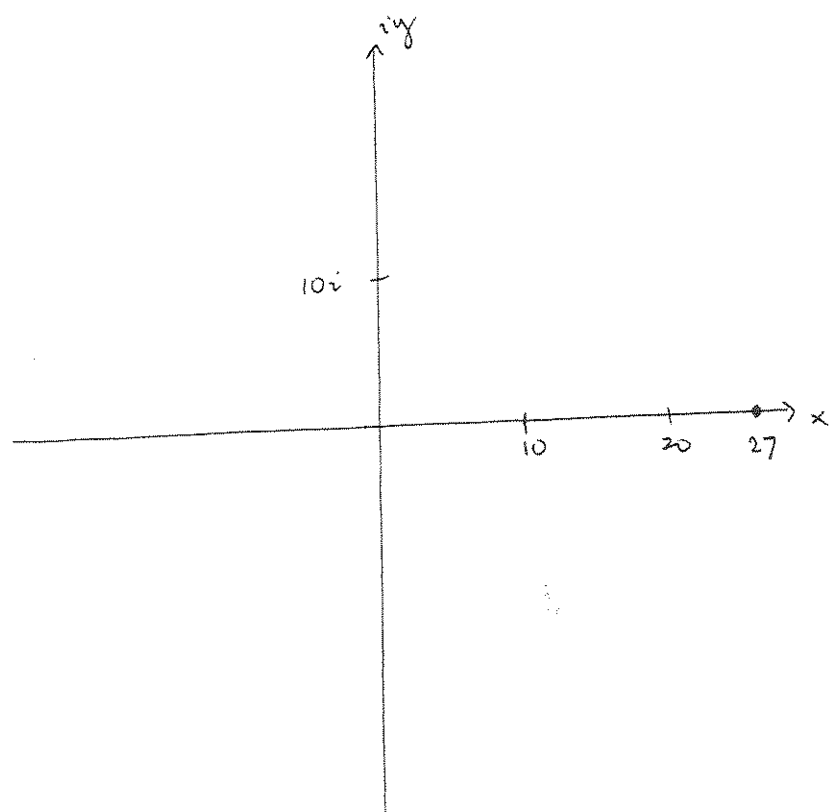
(Euler formula definition that agrees with power series def's)

then $\boxed{e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}}$ (trig addition angle formulas)

So, for $z = |z| e^{i\theta}$, $w = |w| e^{i\phi}$ $|z|$ = modulus of z, $\theta = \arg(z)$

$\boxed{zw = |z||w| e^{i(\theta+\phi)}}$ moduli multiply & arguments add.

<u>Example</u>: Find all solutions to $z^3 = 27$.
Once using polar coordinates (above), once using rectangular coords



About solving polynomial equations in $\mathbb{C}$

1• Every complex number $z_0 \neq 0$ has two $\sqrt{\ }$'s, i.e. solutions z to $z^2 = z_0$ (and they are opposites)

proof 1 : If $z_0 = re^{i\theta}$
let $z = \sqrt{r}\,e^{i\theta/2}$, $\sqrt{r}\,e^{i(\frac{\theta}{2}+\pi)} = -\sqrt{r}\,e^{i\theta/2}$

proof 2 : Write $z_0 = x_0 + iy_0$
$z = x + iy$.

Then $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$ $\quad (\overset{?}{=} z_0 = x_0 + iy_0)$

level set picture

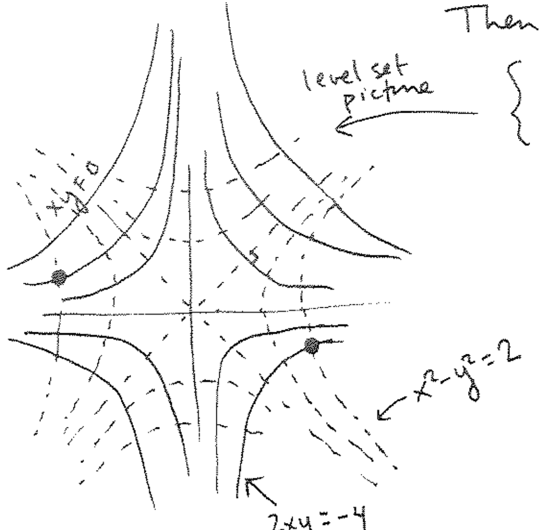
so we want $\left\{ \begin{array}{l} x^2 - y^2 = x_0 \\ 2xy = y_0 \end{array} \right.$

for $x \neq 0$, $y = \dfrac{y_0}{2x}$, subs into 1st eqtn :

$x^2 - \dfrac{y_0^2}{4x^2} = x_0 \rightarrow 4x^4 + (4x_0)x^2 - y_0^2 = 0$

quadratic in x^2, find sol's ...

(also check $y \neq 0$)



$\sqrt{2+4i}$

which proof do you prefer?

2• Every quadratic equation $az^2 + bz + c = 0$ $\quad (a \neq 0)$, $a, b, c \in \mathbb{C}$
has two roots, counting multiplicity

pf with field axioms mimic derivation of quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

, and you know from 1• that you can take $\sqrt{\ }$'s.

3• We'll prove the fundamental theorem of algebra in this course:
(which you've been assuming is true forever, at least for real coef's)

FTA

Thm : The poly. $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ $\quad a_i \in \mathbb{C}$

has exactly n roots, counting multiplicity.

(The question of how to find these roots in practice is interesting.)

4. A special polynomial equation is $z^n = 1$
Its solutions are called the n^{th} roots of unity. (And there are n of them)

Using Euler, and DeMoivre, write
$z = |z| e^{i\theta}$.

solve $|z|^n e^{in\theta} = 1$

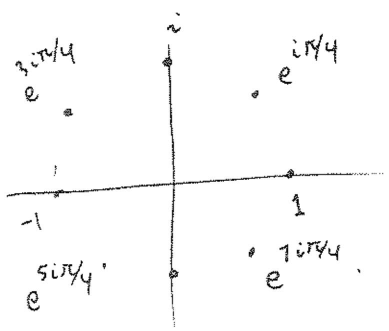
deduce $|z| = 1$

$e^{in\theta} = 1 = e^{i(2\pi k)}$ $\quad k \in \mathbb{Z}$

$n\theta = 2\pi k \quad k \in \mathbb{Z}$

$\theta = \frac{2\pi k}{n}$;

$\theta = 0, \frac{2\pi}{n}, \left(\frac{2\pi}{n}\right)2, \left(\frac{2\pi}{n}\right)3, \dots \frac{2\pi}{n}(n-1)$



the eighth roots of unity,
solt'n's to $z^8 = 1$

Some
Identities & Estimates,
for this analysis class, with a complex twist

$$z = x + iy$$
$$\bar{z} = x - iy$$

1) $\overline{z+w} = \bar{z} + \bar{w}$

2) $\overline{zw} = \bar{z}\,\bar{w}$ $\qquad \overline{(z/w)} = \bar{z}/\bar{w} \quad (w \neq 0)$

3) $|z|^2 = z\bar{z}$

4) $z = \bar{z}$ iff z real
$z = -\bar{z}$ iff z imag.

5) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$
$\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$

6) $\bar{\bar{z}} = z$

7) $|zw| = |z||w|$
$\left(|z/w| = \frac{|z|}{|w|} \quad w \neq 0 \right)$

8) $-|z| \leq \operatorname{Re} z \leq |z|$
$-|z| \leq \operatorname{Im} z \leq |z|$

i.e. $|\operatorname{Re} z| \leq |z|$
$|\operatorname{Im} z| \leq |z|$

9) $|\bar{z}| = |z|$

* 10) $|z + w| \leq |z| + |w|$
Δ inequality

* 11) $|z - w| \geq |z| - |w|$
reverse Δ ineq

* 12) $\left| \sum_{i=1}^{n} z_i w_i \right|^2 \leq \left(\sum |z_i|^2 \right)^{1/2} \left(\sum |w_i|^2 \right)^{1/2}$
complex Cauchy-Schwarz