Math 4200-1
Mon 8/22
JTB 120 11:50–12:40

HW for Friday August 26
1.1 1b, 2c, 4a, 6a, 7, 11 (multiplication properties only), 14, 17a
1.2 1a, 2b, 4, 5, 8, 11, 14, 19

- roll
- organization
- who - background - plans - special

Complex analysis is like Calculus (derivatives and integrals play a major role), except the functions $f(z)$ have complex number domain & range ~ so 1st review the complex plane:

<u>Def</u>: the complex plane $\mathbb{C}$ is defined to be the real vector space $\mathbb{R}^2$ with the additional operation of complex multiplication:

$$\left.\begin{array}{l}\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}:=\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix} \\ \alpha\begin{bmatrix}x_1\\x_2\end{bmatrix}:=\begin{bmatrix}\alpha x_1\\ \alpha x_2\end{bmatrix}\end{array}\quad \forall\, x_i, y_i, \alpha\in\mathbb{R}\right\}\text{ real vector space}$$

$$\begin{bmatrix}x_1\\y_1\end{bmatrix}\begin{bmatrix}x_2\\y_2\end{bmatrix}:=\begin{bmatrix}x_1x_2-y_1y_2\\x_1y_2+y_1x_2\end{bmatrix}\quad\Big\}\text{ complex multiplication}$$

If we identify $x+iy$ with $\begin{bmatrix}x\\y\end{bmatrix}$, then the multiplication axiom reads

$$(x_1+iy_1)(x_2+iy_2):=(x_1x_2-y_1y_2)+i(x_1y_2+y_1x_2)$$

which implies in particular that $i^2=-1$.

$\mathbb{C}$ satisfies the <u>field</u> axioms of Algebra: $\forall z,w,s\in\mathbb{C}$ there holds

$$+\begin{cases} z+w=w+z\\ z+(w+s)=(z+w)+s\\ z+0=z\\ z+(-1z)=0\end{cases}\Bigg\}\text{ these are just (some of the) } \mathbb{R}^2 \text{ vector space axioms, checked in 2270}$$

$$\cdot\begin{cases}\text{(i)}\quad zw=wz\\ \text{(ii)}\quad (zw)s=z(ws)\\ \text{(iii)}\quad 1z=z\\ \text{(iv)}\quad \exists!\, z^{-1}\text{ s.t. } zz^{-1}=1\quad \forall z\neq 0\end{cases}$$

distrib. prop (v) $z(w+s)=zw+zs$

you check these in HW! :)

Further notation:

If $z = x + iy$ $\quad (x, y \in \mathbb{R})$

$Re(z) := x$ "real part of z"

$Im(z) := y$ "imaginary part of z"

$\bar{z} := x - iy$ "conjugate of z"

$|z| := \sqrt{x^2 + y^2}$ "magnitude, or modulus, of z"

Check:

$\overline{zw} = \bar{z}\,\bar{w}$

$|z| = \sqrt{z\bar{z}}$

$$\mathbb{C} \qquad\qquad \mathbb{R}^2$$

$$x + iy \longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

<u>Geometry of $\mathbb{C}$</u>: Since $\mathbb{C}$ is identified with $\mathbb{R}^2$ we represent it via this identification. Then complex addition corresponds to $\mathbb{R}^2$ vector addition, and multiplication becomes very interesting geometrically, via polar coords.

polar coords:

if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$ $\quad r \geq 0$

then $z := x + iy = r\cos\theta + ir\sin\theta$

$|z| = \sqrt{x^2 + y^2} = r$

$\arg(z) := \theta$ (when $r > 0$). θ is uniquely determined up to integer multiples of 2π

"argument of z"

<u>Example</u>

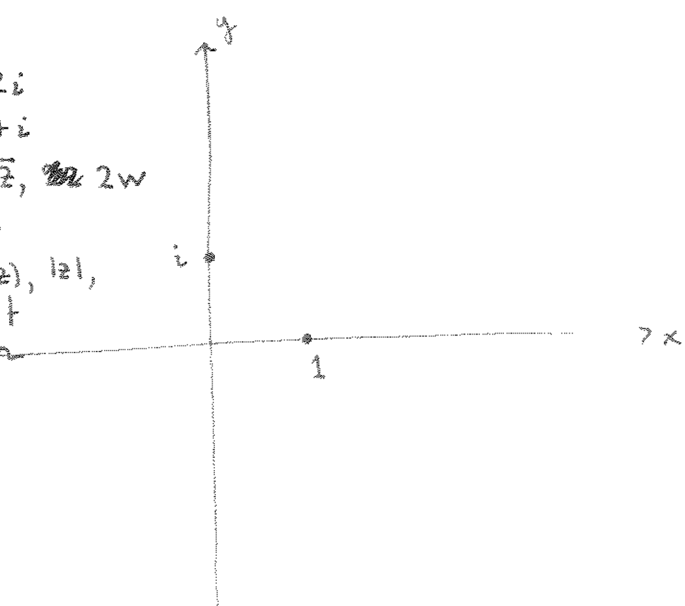
Let $z = 2 - 2i$

$w = \sqrt{3} + i$

1) Plot $z, w, \bar{z}$, $2w$

$z + w$.

2) Find $\arg(z)$, $|z|$, and represent them in your picture



multiplication's geometric meaning:

Write $z = r(\cos\theta + i\sin\theta)$ $\quad r = |z|,\ \theta = \arg z$

$w = \rho(\cos\phi + i\sin\phi)$ $\quad \rho = |w|,\ \phi = \arg w$

Then $zw = r\rho\left(\cos(\theta+\phi) + i\sin(\theta+\phi)\right)$

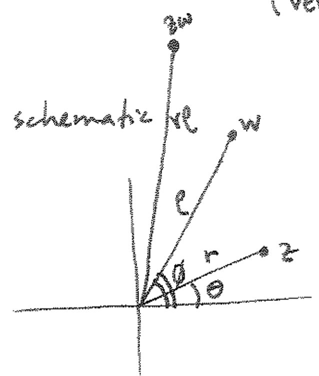
So $|zw| = |z||w|$ (could check w/o polar coords)

$\arg(zw) = \arg z + \arg w$

(very neat!)

Check this!!

schematic



<u>Corollary</u>: DeMoivre's formula

for $n \in \mathbb{N}$ and $z = r(\cos\theta + i\sin\theta)$

$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

pf: by induction

<u>Examples</u> ① Find $\frac{1}{3+4i}$ algebraically & geometrically

② Solve $z^2 = 9i$ alg & geom

③ Find all sol'ns to $z^3 = 27$