Math 4200
Wed Sept 5

For $z = x+iy$, $f(z) = u(x,y) + i v(x,y)$ $\quad$ ($u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$)

Recall

$f'(z) =$

$f_x(z) =$

$f_y(z) =$

Cauchy-Riemann eqtns:

If $f'(z)\ \exists$, <u>then</u> $f' = f_x = -i f_y$

$\boxed{CR}$ $\boxed{\begin{array}{l} \text{i.e. } u_x = v_y \\ \quad\ \ u_y = -v_x \end{array}}$

The converse is true, if we require $u, v \in C^1$:

<u>Theorem</u> let $f(x+iy) := u(x,y) + i v(x,y)$ where the real fcns u, v are C^1 (continuous with continuous first partials) in a nbhd of (x_0, y_0), and satisfy $\boxed{CR}$ at (x_0, y_0).

Then f is complex differentiable at $z_0 = x_0 + i y_0$, with

$$f'(z_0) = u_x + i v_x \quad \text{there.}$$
$$= -i(u_y + i v_y)$$

(you have a HW exercise showing why CR by itself is not enough to get analytic)

<u>proof</u>: Fill this in after digesting page 2, and recalling conditions which imply multivariable functions are differentiable in the analysis (3220) sense.

Differentiability and affine approximation

$z = x+iy$, $z_0 = x_0+iy_0$

$f(z) = u(x,y)+iv(x,y)$

$(x,y)\in\mathbb{R}^2$, (x_0,y_0)

$F(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y)\end{bmatrix}$

Complex approximation lemma

f is differentiable at z_0, with $f'(z_0) = c = a+bi$

iff

$\exists\, c\in\mathbb{C}$ s.t.

$$\boxed{f(z_0+h)-f(z_0) = ch + he(h) \quad \text{with } \lim_{h\to 0} e(h) = 0}$$

pf: $\displaystyle\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = c$

iff

$\displaystyle\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} - c = 0$

so if $f'(z_0)=c$, define $e(h) = \frac{f(z_0+h)-f(z_0)}{h} - c$

so $he(h) = f(z_0+h)-f(z_0)-ch$.

Conversely, if $\exists\, e(h)$ s.t. boxed formula and estimate hold, divide the formula by h, and take $\lim_{h\to 0}$ to deduce $f'(z_0) = h$ ∎

Real version

f is complex differentiable at z_0 with $f'(z_0)=a+bi$

iff

$\exists\, a,b\in\mathbb{R}$ s.t.

$$\boxed{\begin{aligned} &F(x_0+h_1, y_0+h_2) - F(x_0,y_0) \\ &\quad = \begin{bmatrix} a & -b \\ b & a\end{bmatrix}\begin{bmatrix} h_1 \\ h_2\end{bmatrix} + \begin{bmatrix} E_1(h_1,h_2) \\ E_2(h_1,h_2)\end{bmatrix} \\ &\quad\text{where } \frac{\left\|\begin{bmatrix} E_1 \\ E_2\end{bmatrix}\right\|}{\left\|\begin{bmatrix} h_1 \\ h_2\end{bmatrix}\right\|} \to 0 \text{ as } \vec{h}\to 0.\end{aligned}}$$

In other words, iff F is real-differentiable (3220) at (x_0,y_0), with rotation-dilation derivative matrix, i.e. $u_x = v_y = a$, $v_x = -u_y = b$ @ (x_0,y_0)

proof: If $f'(z_0)=a+bi$ $\exists$, then look at complex affine approx lemma. Note $(a+bi)(h_1+ih_2) = ah_1-bh_2 + i(bh_1+ah_2)$.

Interpret in real form:

$$F(x_0+h_1,y_0+h_2)-F(x_0,y_0) = \begin{bmatrix} a & -b \\ b & +a\end{bmatrix}\begin{bmatrix} h_1 \\ h_2\end{bmatrix} + \underbrace{\begin{bmatrix} h_1e_1-h_2e_2 \\ h_1e_2+h_2e_1\end{bmatrix}}_{=\begin{bmatrix} E_1 \\ E_2\end{bmatrix}}$$

this is the real approx. formula, with $\frac{\|\vec{E}\|}{\|\vec{h}\|}\to 0$ as $\vec{h}\to 0$.

Conversely, if real approximation formula holds, then it converts directly to

$$f(z_0+h)-f(z_0) = (a+bi)(h_1+ih_2) + E_1+iE_2$$

define $e(h) := \frac{E_1+iE_2}{h_1+ih_2}$

and deduce $\mathbb{C}$ approx. formula, with $e(h)\to 0$ as $h\to 0$ ∎

page 2 suggests we record: Isomorphism Thm

$$L: \mathbb{C} \to M_{2\times 2}$$

$$L(a+bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

is an algebra isomorphism, i.e

$L(z+w) = L(z) + L(w)$

$L(zw) = L(z)\,L(w)$

L is 1-1 and onto, the rotation-dilation matrices

$$L(\underbrace{(re^{i\theta})\rho e^{i\phi}}_{r\rho e^{i(\theta+\phi)}}) \overset{?}{=} L(re^{i\theta})\,L(\rho e^{i\phi})$$

$$r\rho\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \overset{?}{=} r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \rho \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \quad \checkmark$$

Inverse function theorem

$\mathbb{C}$ | $\mathbb{R}^2$

Let f be complex diff'ble in a neighborhood of z_0, with $f'(z_0) \neq 0$ and $f'(z)$ continuous. Then $\exists$ open sets $z_0 \in \mathcal{U}$, $f(z_0) \in V$ s.t. $f: \mathcal{U} \to V$ is a bijection and $f^{-1}: V \to \mathcal{U}$ is analytic.

Furthermore,
$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}$$

proof

Consider $F(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$

If $f'(z_0) = a+bi \neq 0$ then $[DF](z_0) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ has $\det = a^2+b^2 > 0$, so $[DF]$ non-sing.

f' cont $\Rightarrow u_x, v_x, u_y, v_y$ cont.

Real variables inv fcn thm (3220)

$\Rightarrow \exists$ $(x_0,y_0) \in \mathcal{U}$ open

$(\mathrm{Re}(f(z_0)), \mathrm{Im}(f(z_0)) \in V$ open

s.t. $F: \mathcal{U} \to V$ bijection with C^1 inverse F^{-1}, and

$$[DF^{-1}]_{F(x,y)} = [DF]^{-1}_{(x,y)}$$

so f^{-1} $\exists$ and is analytic, with $(f^{-1})'(f(z)) = \dfrac{1}{f'(z)}$

by isomorphism theorem

Chain rules

$z = x+iy$

$f(z) = u(x,y) + iv(x,y)$

① If f is analytic at z
and g is analytic at $f(z)$,
then $g\circ f$ is analytic at z,
with $\boxed{(g\circ f)'(z) = g'(f(z))\, f'(z)}$

Let $k = f(z+h) - f(z)$
so
$f(z+h) = f(z) + k$.
Then

$$g(f(z+h)) - g(f(z)) = g'(f(z))\, \underset{\substack{\uparrow \\ (f(z+h)-f(z))}}{k} + e_g(k)\,k$$

$\div h$ yields

$$\frac{g(f(z+h)) - g(f(z))}{h} = g'(f(z))\left(\frac{f(z+h)-f(z)}{h}\right) + e_g(k)\frac{k}{h}$$

$\lim\limits_{h\to 0}(\%)$:

$$\lim_{h\to 0} \frac{g(f(z+h)) - g(f(z))}{h} = g'(f(z))\, f'(z)$$

$$+ \lim_{h\to 0} \underset{\substack{\downarrow \\ 0}}{e_g(h)}\ \underset{\substack{\downarrow \\ f'(z)}}{\frac{k}{h}}$$

because $h\to 0 \Rightarrow k\to 0$ $\blacksquare$

② Chain rule for curves

Let $\gamma(t) = x(t) + iy(t)$, $t\in I$, be a differentiable curve,

i.e. $\gamma'(t) = \lim\limits_{\Delta t\to 0} \dfrac{\gamma(t+\Delta t) - \gamma(t)}{\Delta t}$ $\exists$

$(= x'(t) + iy'(t))$

Then

$$\boxed{\underset{\substack{\uparrow \\ \text{real variable}}}{\frac{d}{dt}}(f(\gamma(t))) = \underset{\substack{\uparrow \\ \text{complex deriv.}}}{f'(\gamma(t))}\ \underset{\substack{\uparrow \\ \text{tang. vector at } \gamma(t).}}{\gamma'(t)}}$$

$(x,y) \in \mathbb{R}^2$

$F(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$, if you liked 3220.

or, ① f analytic at z, $f'(z) = a+bi$
iff
F diff'ble at (x,y), $[DF] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

g analytic at $f(z)$, with $g'(f(z)) = c+di$
iff
G diff'ble at $(\operatorname{Re} f(z), \operatorname{Im}(f(z)))$,
$[DG] = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$

3220 Chain rule
$\Rightarrow$ $G\circ F(x,y)$ diff'ble at (x,y)
$[D(G\circ F)] = [DG][DF]$

corresponds to
by isomorphism thm

② Can be proven analogously!

Infinitesimal behavior

The differential map df is defined by $df(\underset{\substack{\uparrow \\ \text{a tangent vector at } \gamma(t)}}{\gamma'(t)}) := \underset{\substack{\uparrow \\ \text{a tangent vector at } f(\gamma(t))}}{\frac{d}{dt} f(\gamma(t))}$

the chain rule for curves says df is a rotation-dilation.

pf: f analytic at $\gamma(t) \Rightarrow$

$$f(\gamma(t+\Delta t)) - f(\gamma(t)) = f'(\gamma(t))\overbrace{(\gamma(t+\Delta t) - \gamma(t))}^{h} + e_f(h)\,h.$$ Divide by Δt, let $\Delta t \to 0$ $\blacksquare$

<u>Example</u> Let $f(z) = \log z = \ln|z| + i \arg z$

Prove $f(z)$ is analytic away from $z=0$ (for any branch choice, i.e. specification of $\arg z$)
and that $(\log z)' = \frac{1}{z}$.
Do this 3 ways! (each is instructive).

1) Inverse fcn thm (+ chain rule).

2) Rectangular CR equations + C^1

3) Polar coord CR eqtns + C^1 → CR equations are equivalent to df (the differential map) being a rotation-dilation.

We can express this in polar coords,
using the <u>chain rule for curves</u>:

$$\begin{cases} \frac{\partial}{\partial r} f(re^{i\theta}) = f'(re^{i\theta}) \cdot e^{i\theta} \\ \frac{\partial}{\partial \theta} f(re^{i\theta}) = f'(re^{i\theta}) \cdot r i e^{i\theta} \end{cases}$$

$$f_\theta = r i f_r$$

$$u_\theta + i v_\theta = r i (u_r + i v_r)$$

$$\boxed{\begin{aligned} u_\theta &= -r v_r \\ v_\theta &= r u_r \end{aligned}}$$