Math 4200
Friday Sept 28

HW: get prepared for the 1^{st} midterm.
(no HW due 10/5)

Exam, Wed next week!

practice exam & sol'ns posted on-line (actual exam from 2005)
all HW sol'ns posted after 5 p.m. today
(they're already up there, except for 2.1–2.3)

Exercise 1 summarizes & foreshadows:

Let $f(z) = \dfrac{z+8}{z^2-4z}$

Let $\gamma(t) = (2+\cos t)e^{2it}$ $0 \le t \le 2\pi$.

Find $\int_\gamma f(z)\,dz$

ints:

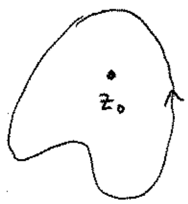
1a) sketch
1b) partial fracts
1c) use deformation thm (homotopy version) for one term, antidiff thm in simply connected domain for the other

§2.4 part 1 : index (or winding number) of a closed path γ w.r.t. $z_0 \in A$

Def

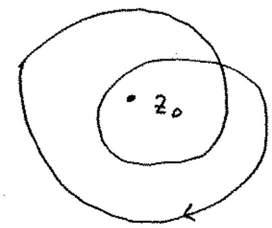
- Index (or a winding number) of a closed path γ (piecewise C^1) wrt z_0 ($z_0 \notin$ range γ) : how many times does γ wind around z_0 in counterclockwise direction?

$I(\gamma; z_0)$



$I(\gamma; z_0) = 1$

$I(\gamma; z_0) = 0$



$I(\gamma; z_0) = -2$

Def: $I(\gamma; z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz$

- $I(\gamma; z_0)$ is a homotopy invariant in $\mathbb{C} \setminus \{z_0\}$ wrt closed curves, by the deformation theorem.

- If $\gamma^n(t) = z_0 + e^{i(2\pi n t)}$ $\quad 0 \le t\ 1 \quad$, $n \in \mathbb{Z}$

  then $\gamma'(t) = 2\pi n i\, e^{i(2\pi n t)}$

  $z - z_0 = e^{i(2\pi n t)}$

  so, $\frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz = \frac{1}{2\pi i} \int_0^1 2\pi n i\, dt = n$ ✓

  (note, we could use $\gamma_r(t) = z_0 + r e^{i(2\pi n t)}$, $r > 0$, as well.
  easy to see directly, and by deformation thm.)

- We show on the next page that any closed path γ with image in $\mathbb{C} \setminus \{z_0\}$ is homotopic to some $\gamma^n(t)$ above, and so has index $n \in \mathbb{Z}$.

<u>Theorem</u> Let $\gamma:[0,1]\to\mathbb{C}\setminus\{z_0\}$ be a closed p.w. C^1 path.
Then γ is homotopic to some γ^n, as on previous page, so that

$$I(\gamma;z_0)=\frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz = n\in\mathbb{Z}.$$

<u>Step 1</u> Translate discussion to origin:

$\gamma_0(t)=\gamma(t)-z_0$ avoids 0, and $\displaystyle\frac{1}{2\pi i}\int_{\gamma_0}\frac{1}{z}\,dz=\frac{1}{2\pi i}\int_\gamma\frac{1}{z-z_0}\,dz$

If $\gamma_0(t)$ is homotopic to the path

$$t\to e^{2\pi i n t}$$

then $\gamma(t)$ is homotopic to $t\to z_0+e^{2\pi i n t}=\gamma^n(t)$

<u>Step 2</u> Let $\gamma_1(t)=\dfrac{\gamma_0(t)}{|\gamma_0(t)|}$ i.e. the projection of γ_0 radially onto the unit circle

γ_1 is homotopic to γ_0 as closed curves in $\mathbb{C}\setminus\{0\}$ by

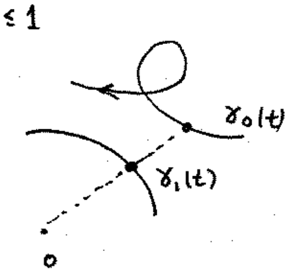
$$H(s,t)=(1-s)\gamma_0(t)+s\,\gamma_1(t)\qquad 0\le s\le 1$$

<u>Step 3</u> Let $\gamma_1(0)=\gamma_1(1)=e^{i\theta}$

Let $\gamma_2(t)=e^{-i\theta}\gamma_1(t)$ (rotate γ_1 by $-\theta$)

Then γ_1 is homotopic to γ_2 as closed curves in $\mathbb{C}\setminus\{0\}$:

$$H(s,t)=e^{-i\theta s}\gamma_1(t)\qquad 0\le s\le 1.$$



<u>Step 4</u> $\gamma_2(0)=\gamma_2(1)=1$. $\gamma_2(t)$ is moving back & forth around $|z|=1$, as $0\le t\le 1$.

<u>If</u> we can find a continuous fcn $\theta(t)$ s.t. $\theta(0)=0$ and

$$\gamma_2(t)=e^{i\theta(t)}$$

then $\theta(t)$ is a choice for $\arg(\gamma_2(t))$, so $\theta(1)=2\pi n$ some $n\in\mathbb{N}$.
since $\gamma_2(1)=1$.

In this case $\gamma_2(t)$ is homotopic to

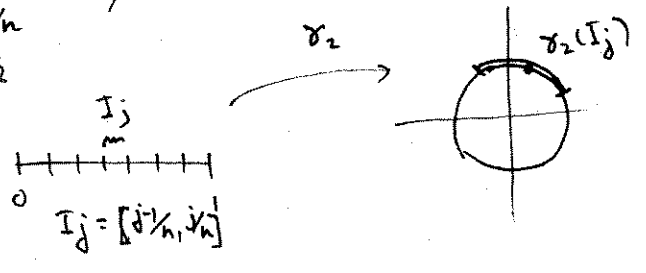
$$t\longrightarrow e^{2\pi i n t}\qquad \text{(as closed curves on } \mathbb{C}\setminus\{0\})$$

via

$$H(s,t)=e^{i[(1-s)\theta(t)+s(2\pi n t)]}\qquad 0\le s\le 1.$$

note, $H(s,0)=H(s,1)=1$

We find $\theta(t)$, continuous on $0\le t\le 1$, by analysis. Pick n s.t. $|t_1-t_2|<1/n$
$\Rightarrow |\gamma_2(t_1)-\gamma_2(t_2)|<\sqrt{2}$



This implies that for $I_j = [\frac{j-1}{n}, \frac{j}{n}]$

$\gamma_2(I_j)$ is (connected) an "interval" of the unit circle spanning $< \pi/2$ radians

$\Rightarrow$ if $\arg(\gamma(\frac{j-1}{n}))$ is chosen, i.e. $\Theta(\frac{j-1}{n})$
then there is a unique continuous choice of $\Theta(t) = \arg(\gamma(t))$
which agrees with this choice for $\Theta(\frac{j-1}{n})$
this gives a value for $\Theta(j/n)$

in this manner
successively define $\Theta(t)$ for $I_1, I_2, \dots I_n$. This yield continuous $\Theta(t)$
$0 \le t \le 1$

<u>Step 5</u> Being homotopic as closed curves ($\cong$) (or, actually, with the same fixed endpoints) in a domain A is <u>transitive</u>:

if $\gamma_0 \cong \gamma_1$ and $\gamma_1 \cong \gamma_2$ then $\gamma_0 \cong \gamma_2$

$H_1(s,t)$ $\quad$ $H_2(s,t)$ $\quad$ $H(s,t) = \begin{cases} H_1(2s,t) & 0 \le s \le 1/2 \\ H_2(2s-1,t) & 1/2 \le s \le 1 \end{cases}$

Thus, our

$\gamma_0 \cong t \to e^{2\pi i n t}$ ■.

(so $\gamma \cong \gamma^{(n)}$)