On Monday we proved:

**Rectangle Lemma**

Let \( R = \{ x + iy \mid a \leq x \leq b, c \leq y \leq d \} \) be a rectangle.

Let \( f: A \to \mathbb{C} \) be analytic. Let \( \gamma = \partial R \) (ccw).

Then \( \oint_{\gamma} f(z)\,dz = 0 \)

and using this lemma we showed:

**Local Antiderivative Theorem:**

\[ f: D(0, r) \to \mathbb{C} \text{ analytic} \]

\[ \Rightarrow \exists F: D(0, r) \to \mathbb{C} \text{ s.t. } F' = f. \]

(Actually, we only used \( f \) continuous + result of rectangle lemma, to get \( F \)！)

Today, combine this result with the notion of homotopy to prove a global antiderivative theorem for simply-connected domains, and to find conditions under which contour integrals for a fixed analytic \( f \) remain unchanged in value w.r.t. change in contour.

**Definition.** Let \( A \subseteq \mathbb{C} \) be open, connected.

Let \( \gamma_0, \gamma_1: [0, 1] \to A \) continuous paths.

Then \( \gamma_0 \) is homotopic to \( \gamma_1 \) in \( A \)

iff

\[ \exists \text{ homotopy } H: \{ (s, t) \mid 0 \leq s \leq 1, 0 \leq t \leq 1 \} \to A \text{ continuous} \]

s.t.

\[ H(0, t) = \gamma_0(t), \quad 0 \leq t \leq 1. \]

\[ H(1, t) = \gamma_1(t). \]
**Special cases of homotopic curves:**

Def: \( \gamma_0, \gamma_1 \) are homotopic with fixed endpoints in \( A \) if

\[
\begin{align*}
\gamma_0(0) &= \gamma_1(0) = P \\
\gamma_0(1) &= \gamma_1(1) = Q
\end{align*}
\]

and \( \exists \) homotopy \( H(s,t) = \gamma_s(t) \) s.t.

\[
\begin{align*}
\gamma_0(0) &= P & \gamma_0(1) &= Q \\
\gamma_1(0) &= P & \gamma_1(1) &= Q
\end{align*}
\]

0 \leq s \leq 1

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Def: let \( \gamma_0, \gamma_1 \) as above, except that they are closed curves,

\[
\begin{align*}
\gamma_0(0) &= \gamma_0(1) = P \\
\gamma_1(0) &= \gamma_1(1) = Q
\end{align*}
\]

Then \( \gamma_0, \gamma_1 \) are homotopic in \( A \) as closed curves if \( \exists \) continuous homotopy \( H: [0,1] \times [0,1] \to A \)

s.t.

\[
H(s,0) = H(s,1) \quad \text{, i.e. each } \gamma_s(t) = H(s,t) \text{ is closed.}
\]

(\( \text{but the terminal = initial endpoint may change with } s \))

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(label segments by their image contour)
Def. A connected open set $A$ is simply connected if every closed curve $\gamma: [0,1] \rightarrow A$ is homotopic to a closed curve to some point $z_0 \in A$, i.e.

$$H: [0,1] \times [0,1] \rightarrow A \text{ continuous}$$

$$H(0,t) = \gamma(t)$$
$$H(1,t) = z_0$$
$$H(s,0) = H(s,1)$$
$$0 \leq s \leq 1$$

Example. A is called star-shaped iff $\exists z_0 \in A$ s.t. $\forall z \in A$, $0 \leq s \leq 1$

$$(1-s)z + sz_0 \in A$$

If $A$ is star-shaped and $\gamma: [0,1] \rightarrow A$

define

$$H(s,t) = (1-s)\gamma(t) + sz_0$$

shrinks $\gamma$ to $z_0$ as $0 \leq s \leq 1$.

so star-shaped domains are simply connected.

Example: The domain for the standard branch of $\log z$, i.e. $C \setminus \{x | x \in \mathbb{R}, x \leq 0\}$

is star-shaped with respect to 1, hence simply connected.

Example: $C \setminus \{0\}$ is not simply connected.

You prove this in HW using complex analysis, in fact using the theorems we prove today!
Homotopy Lemma: Let $A$ open, connected, $f: A \to \mathbb{C}$ analytic.

Let $S = \{(x, y) \mid 0 \leq x, y \leq 1\}$ denote the unit square.

Let $\partial S$ the boundary, oriented c.c.

Let $H: S \to A$ continuous.

Then $T^1 = H(\partial S)$ a p.w. $C^1$ contour.

Then $\int_T f(z) \, dz = 0$.

We will prove this lemma on the next page. It has the following consequences.

Theorem 1: Antiderivative in simply connected domains:

If $A$ is simply connected, $f: A \to \mathbb{C}$ analytic

then $\exists$ antideriv $F: A \to \mathbb{C}, \ F' = f$.

If $\Gamma$ suffices to prove $\oint_{\gamma} f(z) \, dz = 0$ whenever $\gamma$ is p.w. $C^1$ closed in $A$.

For such $\gamma$ consider a homotopy $g \gamma \gamma$ to a fixed pt, thru closed curves.

\[
\oint_{\gamma} f(z) \, dz = \oint_{\gamma_0} f(z) \, dz - \oint_{\gamma_1} f(z) \, dz + \oint_{\gamma_2} f(z) \, dz.
\]

(The technical pt: $\alpha$ might only be continuous, not p.w. $C^1$.

If $\alpha$, look at proof of homotopy lemma on next page to see how to make proof rigorous.)

Theorem 2: Deformation Theorem

Let $A$ open, connected (not nec. simply connected)

$f: A \to \mathbb{C}$ analytic.

If $\gamma_0, \gamma_1$ are p.w. $C^1$ and homotopic in $A$ either with fixed endpoints, or as closed curves

then $\oint_{\gamma_0} f(z) \, dz = \oint_{\gamma_1} f(z) \, dz$.

Proof:

$\gamma_0 \sim \gamma_1$ with fixed endpoints:

$\oint_{\gamma_0} f(z) \, dz = \oint_{\gamma_1} f(z) \, dz + 0$.

$\gamma_0 - \gamma_1$, as closed curves:

$\oint_{\gamma_0} f(z) \, dz - \oint_{\gamma_1} f(z) \, dz + \oint_{\gamma_2} f(z) \, dz$.
Proof of homotopy lemma

Subdivide $S$ into $n^2$ subquadrates $S_{k_1}$ of side-length $\frac{1}{n}$.

$S = \{(x, t), 0 \leq x, t, s \leq 1\}$

$H(DS) = T$

$H(DS_{k_1}) = T_{k_1}$

(replace any arcs of $T_{k_1}$ which are not piecewise $C^1$ with constant speed line segment paths between the vertices.)

By interior cancellation,

$$\int_{T} f(z) \, dz = \sum_{k_1} \int_{T_{k_1}} f(z) \, dz$$

Note

1. $H(S)$ is compact, $CA \Rightarrow \exists \varepsilon > 0$ s.t. $D(z, \varepsilon) \subset A \ \forall \ z \in H(S)$ (positive distance lemma, 5.1.4)

2. $H$ is continuous on $S$, hence uniformly continuous. Thus for $\varepsilon > 0 \in (1)$, $\exists \delta > 0$ s.t. $\|r \cdot (s, t) - (s', t')\| < \delta \Rightarrow |H(s, t) - H(s', t')| < \varepsilon$

3. If $\frac{\partial r}{\partial n} < \varepsilon$ then each $H(S_{k_1}) \subset D(z_{k_1}; \varepsilon) \subset A \ \forall z_{k_1} = H(s_k, t_k)$

4. By local antideriv. then in $D(z_{k_1}; \varepsilon)$, $\int_{T_{k_1}} f(z) \, dz = 0$

Thus, $\int_{T} f(z) \, dz = 0$