

Math 4200

Monday Sept. 24

§2.3

WHERE WE ARE:

We have carefully proven

- If A is a connected open subset of $\mathbb{C}$, $f: A \rightarrow \mathbb{C}$ continuous.
Then contour integrals $\int_{\gamma} f(z) dz$ are path independent
if and only if $\exists$ antiderivative $F(z)$ to $f(z)$.

We have less carefully proven

- If A is open and simply connected, $f: A \rightarrow \mathbb{C}$ analytic and C^1
then $\int_{\gamma} f(z) dz$ are path independent, so $\exists$ antiderivative $F(z)$

issues :- did not define simply connected precisely

- did not show path independence for all p.w. C^1 paths
(assumed they crossed in such a way as to create
a finite # of subdomains on which to apply Green)
- had to assume $f \in C^1$, which turns out to not be necessary.

The goal of §2.3 is to deal with these issues,

and to also prove precise (and stronger) versions of the
deformation theorem, about when you can change the contour curves
of $\int_{\gamma} f(z) dz$ without changing the actual
value of the contour integral.

Along the way we introduce the notion of homotopy, key to
many areas of mathematics, especially algebraic topology

Key step: Local antiderivative theorem

Let $f: D(z_0, r) \rightarrow \mathbb{C}$ be analytic

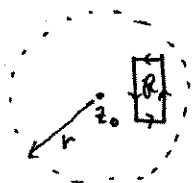
Then $\exists F: D(z_0, r) \rightarrow \mathbb{C}$ s.t. $F' = f$ in D .

This local antiderivative theorem will follow from

Rectangle Lemma: Let $f, D(z_0, r) = D$ as above.

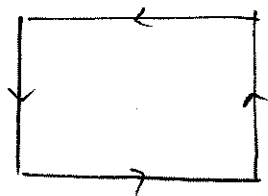
Let $R = [a, b] \times [c, d] \subset D$, i.e. $R = \{x+iy, | a \leq x \leq b, c \leq y \leq d\}$
 $\gamma = \partial R$ (counterclockwise)

Then $\oint_{\gamma} f(z) dz = 0$



Rectangle Lemma proof: (Goursat)

(2)

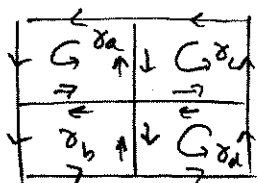


R , diagonal length $:= D$
perimeter $:= P$

$\gamma = \text{p.w. } C^1 \text{ bdry curve, consisting of 4 paths}$

want $\oint_{\gamma} f(z) dz = 0$

subdivide.

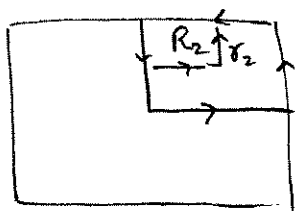


$$\int_{\gamma} f(z) dz = \int_{\gamma_a} + \int_{\gamma_b} + \int_{\gamma_c} + \int_{\gamma_d}$$

$$\Rightarrow \left| \int_{\gamma} f(z) dz \right| \leq |S| + |S| + |S| + |S|$$

$$\leq 4 \left| \int_{\gamma_1} f(z) dz \right|$$

where $\left| \int_{\gamma_1} f(z) dz \right|$ is
the max of the 4 moduli

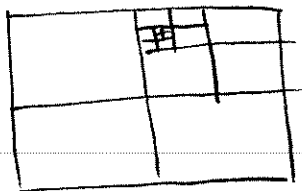


$$\gamma_1 = \partial R_1$$

pick $\gamma_2 = \partial R_2$ s.t.

$$\left| \int_{\gamma_1} f(z) dz \right| \leq 4 \left| \int_{\gamma_2} f(z) dz \right|$$

Induct: $R > R_1 > R_2 > \dots > R_k$



$$\left| \int_{\gamma} f(z) dz \right| \leq 4^k \left| \int_{\gamma_k} f(z) dz \right|$$

$$D_k = \text{diagonal}(R_k) = 2^{-k} D$$

$$P_k = \text{perimeter}(R_k) = 2^{-k} P$$

Since the $\{R_k\}$ are nested, with diameter $\rightarrow 0$

$$\bigcap_{k=1}^{\infty} \text{cl}(R_k) = z_0 \in \text{cl}(R)$$

↑
closure

punchline:

f is analytic at z_0

thus

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + (z-z_0)\varepsilon(z) \quad \varepsilon(z) \rightarrow 0 \text{ as } z \rightarrow z_0$$

Let $\varepsilon > 0$. Pick k s.t. $|\varepsilon(z)| < \varepsilon \quad \forall z \in \bar{R}_k$

$$\begin{aligned} \Rightarrow \int_{\gamma_k} f(z) dz &= \int_{\gamma_k} \underbrace{f(z_0) + f'(z_0)(z-z_0)}_{= f(z_0)z + f'(z_0)\left(\frac{z-z_0}{z}\right)} dz + \int_{\gamma_k} (z-z_0)\varepsilon(z) dz \\ &\quad \parallel \\ &\quad 0 \\ &\quad \text{FTC!! , closed path} \end{aligned}$$

$$\Rightarrow \int_{\gamma_k} f(z) dz = \int_{\gamma_k} (z-z_0)\varepsilon(z) dz$$

$$\Rightarrow |I_0| \leq \int_{\gamma_k} \underbrace{|z-z_0|}_{\leq D_k} \underbrace{|\varepsilon(z)|}_{\leq \varepsilon} \underbrace{|dz|}_{\text{arclength}} \leq \varepsilon D_k P_k = \varepsilon 4^{-k} DP$$

$$\Rightarrow \left| \int_{\gamma} f(z) dz \right| \leq 4^k (\varepsilon 4^{-k} DP) = \varepsilon DP$$

true $\forall \varepsilon > 0$

$$\Rightarrow \left| \int_{\gamma} f(z) dz \right| = 0 \quad \blacksquare, \text{ for Rectangle lemma.}$$

$$\text{(i.e. } \int_{\gamma} f(z) dz = 0 \text{)}$$

now complete the proof of local antiderivative theorem...

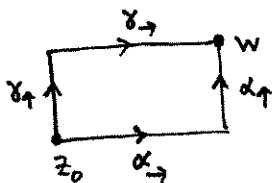
it's a modification of Friday's antideriv. thm, but using only coord-dir. contours:

Let $w \in D(z_0; r)$.

$R(w)$

Consider the closed rectangle which has z_0 and w as opposite corners

(this rect. may be a line segment, if $w - z_0$ is purely real or imag.)



Note $R(w) \subset D(z_0; r)$

Define $F(w) := \int_{\gamma_1 + \gamma_2} f(z) dz$ (move horiz, then vert, from z_0 to w , around $\partial R(w)$)

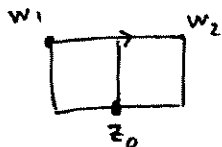
$$= \int_{\gamma_1 + \gamma_2} f(z) dz \quad (\text{vert, then horiz.})$$

by Rect. Lemma.

Consequence: If w_1, w_2 differ by a pure real, or pure imaginary number

then

$$F(w_2) - F(w_1) = \int_{w_1 \rightarrow w_2} f(z) dz \quad (\text{horizontal contour})$$



$$\alpha_2 = \int_{w_1 \rightarrow w_2} f(z) dz \quad (\text{vertical contour})$$

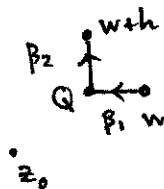
Finally, for $w \in D(z_0; r)$, $|h| < r - |w|$,

consider

$$F(w+h) - F(w) =$$

$$F(w+h) - F(Q) + F(Q) - F(w)$$

$$= \int_{\beta_1 + \beta_2} f(z) dz, \text{ by consequence}$$



$$Q := \operatorname{Re}(w+h) + i \operatorname{Im}(w)$$

$$\begin{aligned} \text{Thus } F(w+h) - F(w) &= \int_{\beta_1 + \beta_2} f(z) dz = \int_{\beta_1 + \beta_2} f(w) dz + \int_{\beta_1 + \beta_2} (f(z) - f(w)) dz \\ &= f(w) z \Big|_w^{w+h} + \text{error} \end{aligned}$$

$$F(w+h) - F(w) = f(w)h + \text{error}$$

$$|\text{error}| \leq \int_0^1 |f(z) - f(w)| dz \leq 2|h| \max_{|z-w| < 2|h|} |f(z) - f(w)|$$

upper bd for curve length

thus $\frac{|\text{error}|}{|h|} \rightarrow 0$ as $h \rightarrow 0$
thus $F'(w) = f(w)$