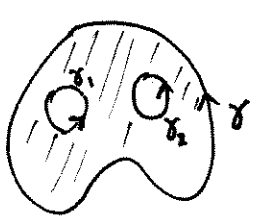
Math 4200
Wednesday 19 Sept.

Finish pages 3–4 Monday.

To finish Theorem 2 "proof" will take some extra room.

Note, while proving Theorem 2, and appealing ~~th~~ to Green's Theorem (for domains with holes), we actually proved:

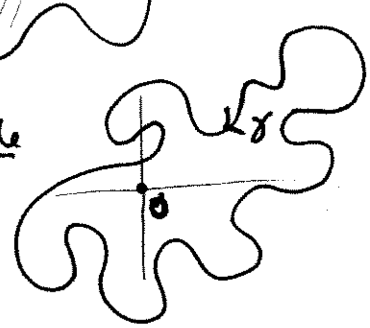
<u>Theorem</u> Let $f(z)$ be analytic and $C^1$ on an open region containing a compact subset $A$, whose boundary is a union of piecewise $C^1$ curves. Let $\gamma$ be the outer boundary curve, with $\gamma_1, \ldots$ the inner bdry curves. Orient curves as indicated




Then $\displaystyle\int_\gamma f(z)\,dz = \sum_i \int_{\gamma_i} f(z)\,dz$

(notice we changed the orientation of the $\gamma_i$ from Greens)

<u>Example</u>



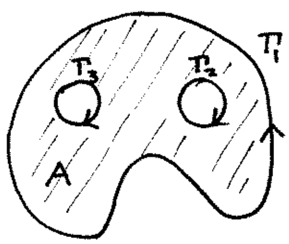

$\displaystyle\int_\gamma \frac{1}{z}\,dz =$

http://en.wikipedia.org/wiki/Stokes'_theorem

## Green's Theorem

(This is just one of the vector calculus "FTC"'s, and in fact one can understand all of them as special cases of a general theorem) called Stokes' Theorem

Let $\langle P(x,y), Q(x,y)\rangle$ be a vector field, $C^1$ on an open domain containing the set $A$ and its boundary $\partial A$.
Orient $\Gamma = \partial A$ so that $A$ is "on the left" as you traverse $\Gamma$:



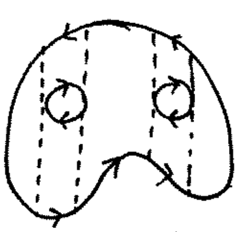

then
$$\boxed{\oint_{\partial A} P\,dx + Q\,dy = \iint_A (Q_x - P_y)\, dA}$$

proof: ① $\displaystyle\oint_{\partial A} P\,dx = \iint_A -P_y\, dA$ , ② $\displaystyle\oint_{\partial A} Q\,dy = \iint_A Q_x\, dA$

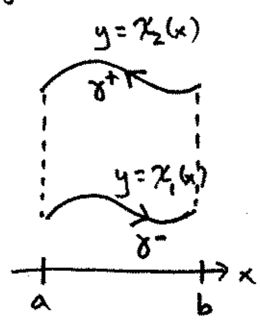
(unfortunately, I have used $A$ for the region, and $dA$ for $dx\,dy$ — these uses of "A" are entirely independent)

① + ② = Green's Thm

① Chop up $A$ into "vertical simple" subregions:



each region:

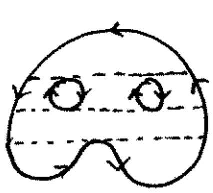


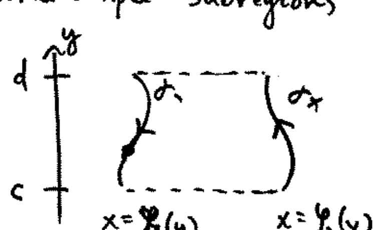

iterate the area integral:

$$\int_a^b \int_{\chi_1(x)}^{\chi_2(x)} -P_y(x,y)\,dy\,dx$$
$$= \int_a^b -P(x,y)\Big]_{y=\chi_1(x)}^{y=\chi_2(x)} dx$$
$$= \int_a^b -P(x,\chi_2(x))\,dx + P(x,\chi_1(x))\,dx$$
$$= \int_{\gamma^+} P\,dx + \int_{\gamma^-} P\,dx$$

add these identities for each region, and use the fact that double integrals and line integrals are additive with respect to partitioning.
Deduce $\displaystyle\iint_A -P_y\,dA = \oint_{\partial A} P\,dx$ ■

② Chop $A$ into "horizontal simple" subregions






$$\int_c^d\int_{\varphi_1(y)}^{\varphi_2(y)} Q_x\,dx\,dy = \int_c^d Q(x,y)\Big]_{x=\varphi_1(y)}^{\varphi_2(y)} dy$$
$$= \int_c^d Q(\varphi_2(y),y)\,dy - Q(\varphi_1(y),y)\,dy$$

then add, as in ①.

$$= \int_{\sigma_2} Q\,dy - \int_{\sigma_1} Q\,dy$$