Math 4200
Wednesday 19 Sept.

Finish pages 3-4 Monday.

To finish Theorem 2 "proof" will take some extra room.

Note, while proving Theorem 2, and appealing to Green's Theorem (for domains with holes), we actually proved:

**Theorem** Let \( f(z) \) be analytic and \( C^1 \) on an open region containing a compact subset \( A \), whose boundary is a union of piecewise \( C^1 \) curves. Let \( \gamma \) be the outer boundary curve, with \( \gamma_i \) the inner boundary curves. Orient curves as indicated.

Then \( \int_{\gamma} f(z) \, dz = \sum_i \int_{\gamma_i} f(z) \, dz \)

(Notice we changed the orientation of the \( \gamma_i \) from Greens)

**Example**

\( \int_{\gamma} \frac{1}{z} \, dz = \)
Green's Theorem

(This is just one of the vector calculus "FTC"s, and in fact one can understand all of them as special cases of a general theorem, called "Stokes' Theorem".)

Let \( \langle P(x,y), Q(x,y) \rangle \) be a vector field, \( C \) on an open domain containing the set \( A \) and its boundary. Orient \( T = \partial A \) so that \( A \) is "on the left" as you traverse \( T \):

\[
\oint_{\partial A} P\,dx + Q\,dy = \iint_A (Q_x - P_y)\,dA
\]

\begin{align*}
\text{proof:} & \quad 1. \quad \oint_{\partial A} P\,dx = \iint_A -P_y\,dA, \quad 2. \quad \oint_{\partial A} Q\,dy = \iint_A Q_x\,dA \\
& \quad (\text{unfortunately, I have used } A \text{ for the region, and } dA \text{ for } dx, dy -- \text{ these use of } "A" \text{ are entirely independent})
\end{align*}

1. Chop up \( A \) into "vertical simple" subregions:

Chop up the area integral:

\[
\iint_A (Q_x - P_y)\,dA = \int_a^b \int_{\gamma_2(x)}^{\gamma_1(x)} (-P_y\,dy)\,dx
\]

\[
= \int_a^b \pi(x)\,dx
\]

\[
= \int_a^b -P(x,\gamma_2(x))\,dx + P(x,\gamma_1(x))\,dx
\]

\[
= \int_a^b P\,dx + \int_{\gamma_1}^{\gamma_2} P\,dx
\]

Deduce \( \iint_A -P_y\,dA = \iint_A P\,dx \)

2. Chop \( A \) into "horizontal simple" subregions:

\[
\iint_A (Q_x - P_y)\,dA = \int_c^d \int_{\gamma_3(y)}^{\gamma_2(y)} (Q_x - P_y)\,dx\,dy
\]

\[
= \int_c^d Q_{y_2(y)}\,dy - Q_{y_1(y)}\,dy
\]

\[
= \int_c^d Q\,dy - \int_{y_1(y)}^{y_2(y)} Q\,dy
\]

then add, as in 1.