Math 4200
Monday 17 Sept.

If $\gamma : [a,b] \to \underset{\mathbb{C}}{\overset{\cap}{A}}$ open, $\gamma \in C^1$ and $f : A \to \mathbb{C}$ is continuous then $\int_\gamma f(z)\,dz =$

$\int_\gamma |f(z)|\,|dz| =$

and FTC states:

integral estimate:

<u>Complete exercise 6 Friday:</u>

for $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, $f(z) = z^n$, $n \in \mathbb{Z}$

compute $\int_\gamma f(z)\,dz$

Note $n=1$ is special (Exercise 3 Fri.)
Can you do this case with FTC?

discuss <u>reparameterization</u>, <u>contour integrals</u> $\leftrightarrow$ <u>real line integrals</u>, page 4 Friday.

Contour curve algebra

Let $\gamma : [a,b] \to A$ open, $\gamma \in C^1$.
$\qquad\qquad\qquad\qquad\quad \cap$
$\qquad\qquad\qquad\qquad\quad \mathbb{C}$

Definition: $-\gamma : [a,b] \to A$ is the curve $-\gamma(t) := \gamma(b - (t-a)) = \gamma(a+b-t)$, $\quad a \le t \le b$

i.e. γ traversed in the reverse direction.

By the reparameterization theorem,

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

Now, consider piecewise C^1 contours:

Recall, we defined $\gamma = [\gamma_1, \gamma_2 \dots \gamma_n]$ to be piecewise C^1 if each $\gamma_j : [a_j, b_j] \to \mathbb{C}$ is C^1, and $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ $\quad j = 1, \dots n-1$

As well, defined
$\gamma_1(a_1)$ to be the initial point of γ,
$\gamma_n(b_n)$ to be the terminal point of γ

Note: our text actually requires $b_j = a_{j+1}$, so that γ is continuous on the interval $[a_1, b_n]$, and C^1 on each $[a_j, b_j]$.

If $\gamma = [\gamma_1, \gamma_2, \dots \gamma_n]$ is piecewise C^1 in our sense (which includes the text's) we write

$$\boxed{\gamma = \gamma_1 + \gamma_2 + \dots \gamma_n}$$

and define $-\gamma = [-\gamma_n, -\gamma_{n-1}, \dots -\gamma_1]$, i.e.

$$\boxed{-\gamma = -\gamma_n - \gamma_{n-1} - \dots - \gamma_1}$$

$$\boxed{\int_{\gamma} f(z)\,dz = \int_{\gamma_1+\gamma_2+\dots+\gamma_n} f(z)\,dz := \sum_j \int_{\gamma_j} f(z)\,dz}$$

Theorem Let $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$ be piecewise C^1, with range in $A \subset \mathbb{C}$, A open. $f: A \to \mathbb{C}$ continuous. Then

(1) $\displaystyle\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$ pf: $\displaystyle\int_{-\gamma_j} f(z)\,dz = -\int_{\gamma_j} f(z)\,dz$. now sum over j ■

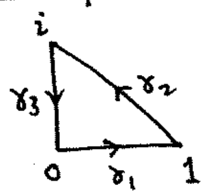
(2) If $\exists$ antideriv. $F: A \to \mathbb{C}$ with $F' = f$

then $\displaystyle\int_{\gamma} f(z)\,dz = F(Q) - F(P)$ where P, Q are the initial, terminal points of γ

pf: $$\int_{\gamma} f(z)\,dz = \sum_j \int_{\gamma_j} f(z)\,dz$$
$$= \sum_j F(\gamma(b_j)) - F(\gamma(a_j))$$
$$= F(\gamma(b_n)) - F(\gamma(a_1))$$ because the series telescopes, $\gamma(b_j) = \gamma(a_{j+1})$, $j = 1, \dots n-1$ ■

(3) $$\left|\int_{\gamma} f(z)\,dz\right| \le \sum_j \int_{\gamma_j} |f(z)||dz| := \int_{\gamma} |f(z)||dz|.$$ ■

Examples



$\gamma = \gamma_1 + \gamma_2 + \gamma_3$ (the particular parameterizations don't matter, just the directions).

	parameterization	Green's thm!	
$\displaystyle\int_{\gamma} 1\,dz$			FTC:
$\displaystyle\int_{\gamma} z\,dz$			FTC:
$\displaystyle\int_{\gamma} \bar{z}\,dz$			no FTC!

$f: A \to \mathbb{C}$ continuous, A open and connected
$A \subset \mathbb{C}$

When does f have an antiderivative $F(z)$, i.e. $F'(z) = f(z)$ $\forall z \in A$?

Theorem 1: The following are equivalent, for $f: A \to \mathbb{C}$ continuous, A open & connected

(i) $\exists\ F: A \to \mathbb{C}$ s.t. $F'(z) = f(z)$ $\forall z \in \mathbb{C}$, (and F is unique up to a constant)

↑ proved before, since if F, G are antiderivs, then $(F-G)' \equiv 0$ on A $\Rightarrow F-G = \text{const}$.

(ii) $\forall$ choices of initial pt P & terminal pt Q in A

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

whenever γ_0, γ_1 both start at P and end at Q (γ_0, γ_1 piecewise C^1)

(iii) $\forall$ piecewise C^1 curves γ which have the same initial and terminal point (:= closed curves γ),

$$\int_{\gamma} f(z)\,dz = 0$$

pf $(i) \overset{\text{FTC}}{\Rightarrow} (iii) \overset{\text{logic}}{\Rightarrow} (ii) \Rightarrow (i)$

↑ use contour integral to define antiderivative

Theorem 2: If A is open and simply connected $f: A \to \mathbb{C}$ analytic and C^1 then $\exists\ F: A \to \mathbb{C}$ s.t. $F'(z) = f(z)$ $\forall z \in A$

"pf" (use Green's thm to verify (iii) above)