HW for Fri 9/20

2.1 2ac, 3, 5, 10, 11, 13, 14

2.2 1ad, 2 (prove with FTC!), 3, 4, 6, 8, 11.

Math 4200
Fri 9/13

Chapter 2: Complex integration

- leads to Cauchy Integral formula and magic theorems:
  - Liouville: bounded entire functions are constant
  - F.T. Algebra: every degree n polynomial has n (complex) roots (counting multiplicity)
  - magic ways to calculate integrals (contour integration)

<u>Definition A</u>: For $f: \underset{\mathbb{R}}{[a,b]} \to \mathbb{C}$ continuous, $f(t) = u(t) + iv(t)$ $\quad u = \operatorname{Re} f$, $v = \operatorname{Im} f$

then

(A1) $$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

is equivalent to

(A2) $$\int_a^b f(t)\,dt := \lim_{\|P\| \to 0} \sum f(t_j^*)\Delta t_j$$

(norm of partition approaches zero)

P: $a = t_0 < t_1 < \ldots < t_n = b$

$t_{j-1} \le t_j^* \le t_j$; $\Delta t_j = t_j - t_{j-1}$

$\|P\| = \max\limits_j \Delta t_j$

$$= \lim_{\|P\| \to 0} \sum \left(u(t_j^*) + iv(t_j^*)\right)\Delta t_j$$

$$= \lim_{\|P\| \to 0} \sum u(t_j^*)\Delta t_j + i\sum v(t_j^*)\Delta t_j$$

$$= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

✓

<u>Exercise 1</u> $$\int_0^{\pi/2} -2\sin t\cos t + i(\cos^2 t - \sin^2 t)\,dt =$$

3) <u>Fundamental Theorem of Calculus</u> (apply real FTC to real & imag parts)

$$\int_a^b f(t)\,dt = F(b) - F(a) \qquad \text{if } F'(t) = f(t) \ \forall t \in [a,b].$$

4) <u>Integral estimate</u>

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$$

pf: apply A2: $\left|\int_a^b f(t)dt\right| = \left|\lim\limits_{\|P\| \to 0} \sum f(t_j^*)\Delta t_j\right|$

$= \lim\limits_{\|P\| \to 0} \left|\sum f(t_j^*)\Delta t_j\right| \underset{\Delta \text{ ineq!}}{\le} \lim\limits_{\|P\| \to 0} \sum |f(t_j)|\Delta t_j$

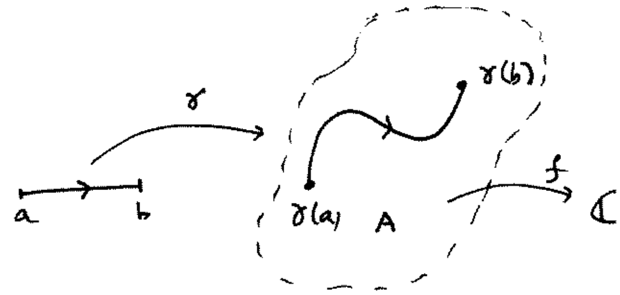
$= \int_a^b |f(t)|\,dt$ $\blacksquare$

<u>Definition B</u> Let $A \subset \mathbb{C}$ open, $f: A \to \mathbb{C}$ continuous (not necessarily analytic, although later on this will be our primary focus)

$\gamma : [a,b] \to A$ a C^1 curve
$[a,b] \subset \mathbb{R}$

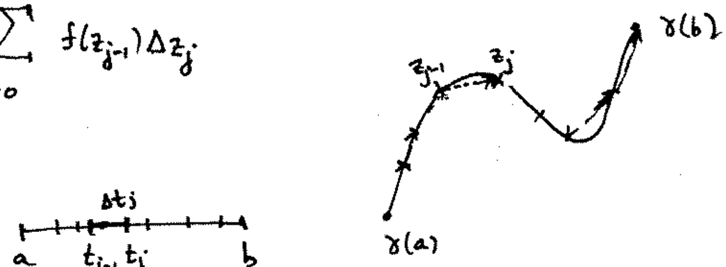
(B1) $$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

(so, you substitute $z = \gamma(t)$ and the differential $dz = \gamma'(t)dt$ to evaluate the <u>contour integral</u>)



in case $|\gamma'(t)| > 0$ on $[a,b]$ (so $|\gamma'(t)| \geq \delta > 0$), then (B1) is equivalent to

(B2) $$\int_\gamma f(z)\,dz = \lim_{\max|\Delta z_j| \to 0} \sum f(z_{j-1})\Delta z_j$$



$$= \lim_{\max|\Delta t_j| \to 0} \sum f(\gamma(t_{j-1})\gamma'(t_{j-1})\Delta t_j + f(\gamma(t_{j-1}))\varepsilon(t_j)\Delta t_j$$

$$\Delta z_j = z_j - z_{j-1} = \gamma(t_j) - \gamma(t_{j-1}) = \gamma'(t_{j-1})\Delta t_j + \varepsilon(t_j)\Delta t_j$$

$$= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt + 0 \quad \checkmark$$

where $\varepsilon(t) \to 0$ uniformly as $\Delta t \to 0$, because $\gamma \in C^1$

<u>Exercise 2</u> Let $\gamma(t) = e^{it}$ $0 \leq t \leq \pi/2$

$f(z) = z$

compute $\int_\gamma f(z)\,dz$

<u>Exercise 3</u> $\gamma(t) = e^{it}$ $0 \leq t \leq 2\pi$

$f(z) = \frac{1}{z}$

compute $$\int_\gamma \frac{1}{z}\,dz$$

(B3) FTC : If $\exists\ F : A \to \mathbb{C}$ analytic, with $F' = f$ on A,

then $$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

pf: $$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$
$$= \int_a^b \underbrace{F'(\gamma(t))\gamma'(t)}_{\frac{d}{dt}F(\gamma(t))}dt = F(\gamma(t))\Big]_a^b \quad \text{by (A3)}$$

(B4) Integral estimate :
$$\left| \int_\gamma f(z)\,dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\,dt \right|$$
$$\leq \int_a^b |f(\gamma(t))\gamma'(t)|\,dt \qquad \text{(A4)}$$
$$= \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

(B5): $$\int_\gamma |f(z)|\,|dz| := \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

so (B4) reads $$\left| \int_\gamma f(z)\,dz \right| \leq \int_\gamma |f(z)|\,|dz|$$

Exercise 4 Rework Exercise 2 using FTC

Exercise 5 What is $\int_\gamma |f(z)|\,|dz|$ for Exercise 2 Example?

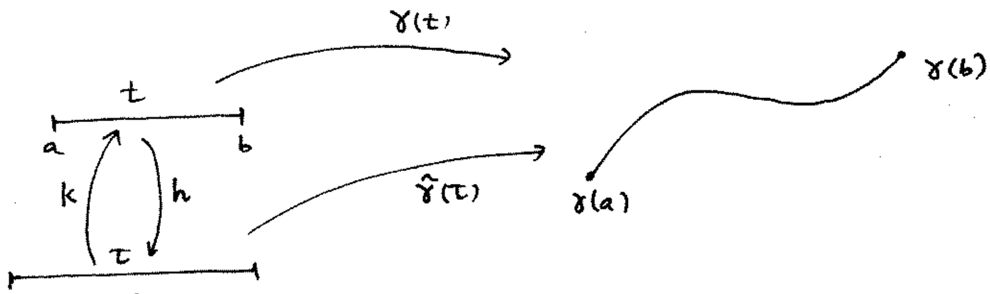
Exercise 6 : Let $\gamma(t) = e^{it}$ $\quad 0 \leq t \leq 2\pi$

$f(z) = z^n \quad n \in \mathbb{Z}.$

What is $\int_\gamma f(z)\,dz$?

Reparameterization: If you reparameterize γ in the same direction the value of the contour integral stays the same. If you reverse direction you get the opposite value

pf:



change of variables

$\tau = h(t)$

$t = k(\tau)$

$$\tilde{\gamma}(\tau) = \gamma(k(\tau)) = \gamma(t)$$

$$\int_{\tilde{\gamma}} f(z)\,dz = \int_c^d f(\tilde{\gamma}(\tau))\,\tilde{\gamma}'(\tau)\,d\tau$$

subs $\tau = h(t)$, $d\tau = h'(t)\,dt$

$$\int_{k(c)}^{k(d)} f(\gamma(t))\,\tilde{\gamma}'(h(t))\,h'(t)\,dt = \begin{cases} +\int_\gamma f(z)\,dz & \text{if } k(c)=a,\ k(d)=b \\ -\int_\gamma f(z)\,dz & \text{if } k(c)=b,\ k(d)=a \end{cases}$$

$$\frac{d}{dt}\tilde{\gamma}(h(t)) = \frac{d}{dt}\gamma(t) \,!$$

$\blacksquare$

Relation of contour integrals to (real) line integrals:

write $f(z) = u(x,y) + i v(x,y)$

$\gamma(t) = x(t) + i y(t)$

then

$$\int_\gamma f(z)\,dz = \int_a^b \left[u(x(t),y(t)) + i v(x(t),y(t))\right]\left[x'(t) + i y'(t)\right] dt$$

$$= \int_a^b u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t)\,dt$$

$$+ \; i \int_a^b v(x(t),y(t))\,x'(t) + u(x(t),y(t))\,y'(t)\,dt$$

$$= \int_\gamma u\,dx - v\,dy + i \int_\gamma v\,dx + u\,dy$$

is to derive formally by writing

$f = u + iv$, $dz = dx + i\,dy$

$$\int_\gamma (u+iv)(dx + i\,dy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy. \quad \checkmark$$