Math 4200
Wed 12 Sept

§ 1.6 : The zoo of common analytic functions, their derivatives;
and branches of their inverses, and of compositions involving their inverses.

(we may still wish to do the conjugate construction on page 3 Monday)

Def: If $f : \mathbb{C} \to \mathbb{C}$ is analytic $\forall z \in \mathbb{C}$, f is called entire

examples:

$f(z) = z^n \quad (n = 0, 1, 2, \ldots)$

$f'(z) = n z^{n-1}$ (by binomial thm or product rule induction)

$f(z) = e^z$

$f'(z) = e^z$ (by direct computation, $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$
or by Cauchy Riemann + C^1 partials)

$f(z) = \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

$f'(z) =$ (chain rule)

$f(z) = \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$f'(z) =$

etc.

"Branches" of analytic functions : overview

If f is entire then it turns out (Picard's theorem) that if f is not a constant function then, in fact, f is almost onto (the precise theorem is that if the range of f omits 2 or more points, then f is constant → called Picard's thm.) furthermore, the zeroes of $f'(z)$ are isolated, so f has local inverse fcns at all but a countable set of z's.

In this case one can construct an "inverse" function g, which satisfies half of the inverse function condition, $f(g(z)) = z$, on a connected open domain A which is $\mathbb{C}$ with a finite number of curves removed. These curves are called branch cuts, and the choice of g is called a branch of the inverse "function". Branch cuts always terminate at ∞ or at a finite value branch point, $f(z)$, where $f'(z) = 0$.

examples (revisited)

$f(z) = z^2$

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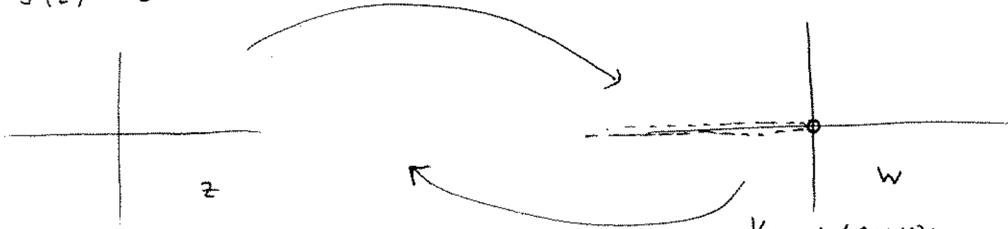


$g(w) = |w|^{1/2} e^{i\left(\frac{\arg w}{2}\right)}$

(= one of the $\sqrt{w}$'s)

$-\pi < \arg w < \pi$

branch cut along the negative real ray.

(but you could use any reasonable curve from 0 to ∞ as the branch cut, and define a different branch of $\sqrt{w}$)

$\text{range}(g) = \{z \mid \operatorname{Re} z > 0\}$
$= 1^{st} \cup 4^{th}$ quadrants

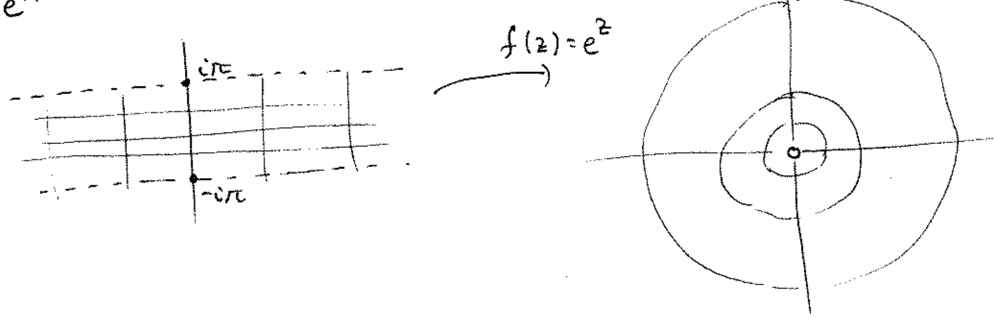
since $f(g(w)) = w$

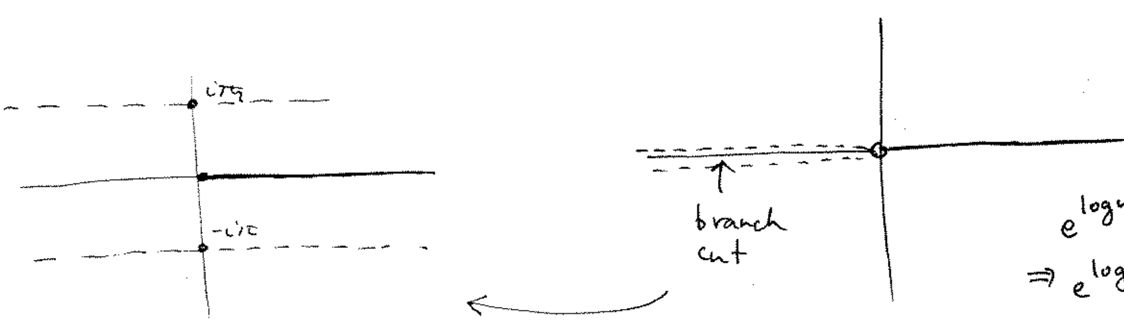
$f'(g(w))\, g'(w) = 1$

$g'(w) = \frac{1}{2\, g(w)}$

$(\sqrt{w})' = \frac{1}{2} w^{-1/2}$, would work for any branch choice, as long as $\sqrt{w}$ is defined consistently

$f(z) = e^x$





$e^{\log w} = w$

$\Rightarrow e^{\log w} (\log w)' = 1$

$(\log w)' = \frac{1}{w}$
for any branch choice.

$g(w) = \log w$
$= \ln|w| + i \arg w$
$-\pi < \arg w < \pi$
"standard branch"

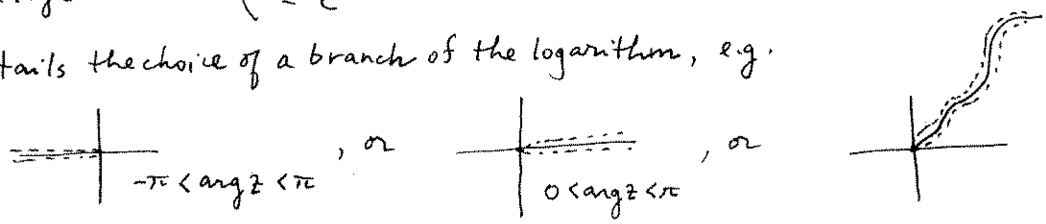
note, on positive x-axis restricts to $\ln x$, inverse of e^x

we kind of had this discussion before, except for computing the derivatives...

for any $a \in \mathbb{C}$

$z^a := e^{a \log z}$ $\quad \left(= e^{a(\ln|z| + i \arg z)} \right)$

entails the choice of a branch of the logarithm, e.g.



$$\frac{d}{dz} z^a = \frac{d}{dz} e^{a \log z} = e^{a \log z} \frac{a}{z} \quad \text{chain rule}$$
$$= z^a \frac{a}{z} = a z^{a-1}$$

(provided we use the same branch of logarithm).

(1) if we use standard branch of log, $\arg x = 0$ $(x \in \mathbb{R}^+)$ $\Rightarrow x^a = e^{a \ln x}$ agrees with Calc def.

(2) if $n \in \mathbb{Z}$, $z^n := e^{n \log z} = e^{n(\ln|z| + i \arg z)} = |z|^n e^{in\theta} = z^n$ algebraic def.

(3) if $n \in \mathbb{N}$, $z^{\frac{1}{n}} := e^{\frac{1}{n} \log z} = e^{\frac{1}{n}(\ln|z| + i \arg z)}$
$= |z|^{1/n} e^{i\frac{\theta}{n}}$ is one of the n^{th} roots of z, depending on branch of log.

<u>example</u> find a branch of $\sqrt{e^z+1}$, and its deriv.

$f(z) = e^z + 1$

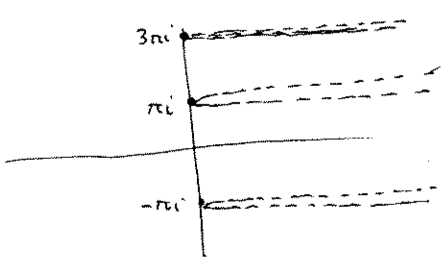
$g(w) = \sqrt{w}$ ⟵ 1st choose a branch of this

then study f-preimages of the branch cut you chose

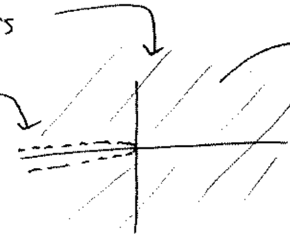
i.e. $f(z)$ a non-pos real

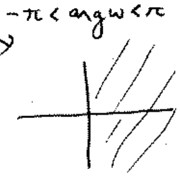
$g(w) = \sqrt{w} = |w|^{\frac{1}{2}} e^{i \frac{\arg w}{2}}$

$-\pi < \arg w < \pi$



$f(z) = e^{z+1}$





If you change this branch, then you change this one too!

$e^z + 1 \notin (-\infty, 0]$

$e^z \notin (-\infty, -1]$

$e^{x+iy} \notin (-\infty, -1]$

$\begin{cases} y \neq \pi + 2\pi k \quad k \in \mathbb{Z} \\ \text{and } e^x \geq 1 \\ \quad x \geq 0 \end{cases}$

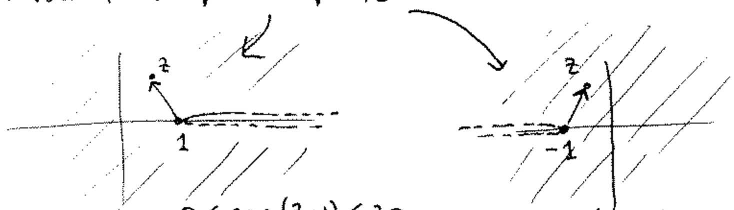
$$\frac{d}{dz}(e^z+1)^{\frac{1}{2}} = \frac{1}{2}(e^z+1)^{-\frac{1}{2}} e^z$$

by chain rule and previous example.

<u>example</u> find branch(es) of

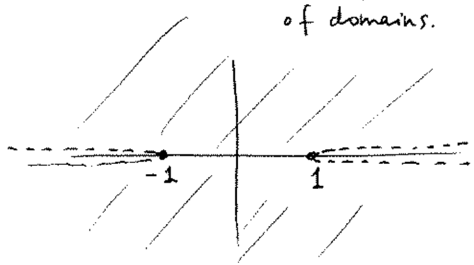
$$f(z) = \sqrt{z^2-1}$$

(1) write it as $\sqrt{z-1}\,\sqrt{z+1}$



$0 < \arg(z-1) < 2\pi$ $\qquad$ $-\pi < \arg(z+1) < \pi$

so product is defined on intersection of domains.



so

$$f(z) = \sqrt{z-1}\,\sqrt{z+1} = \left(|z-1|\,|z+1|\right)^{1/2} e^{i\frac{1}{2}\left(\arg(z-1) + \arg(z+1)\right)}$$

$\arg(z-1) \in (0, 2\pi)$, $\arg(z+1) \in (-\pi, \pi)$

or,

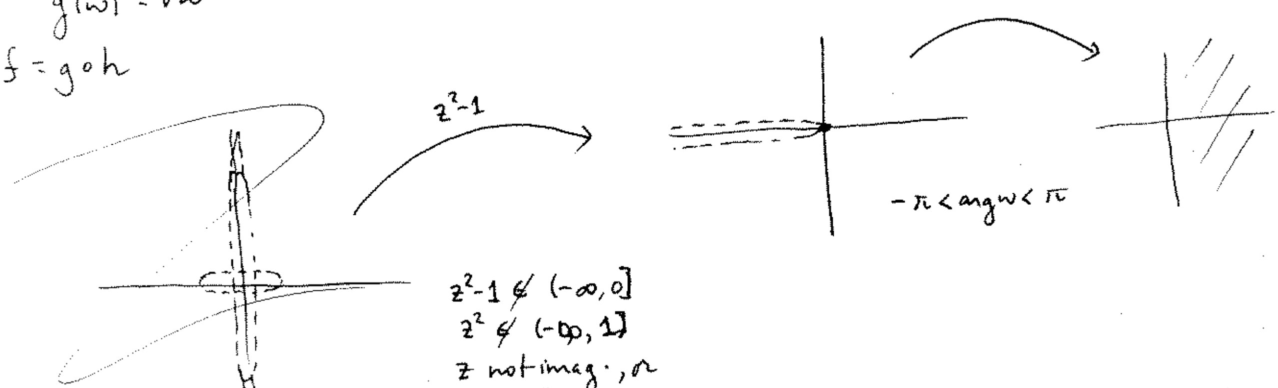
(2) Use method of previous example

$h(z) = z^2 - 1$

$g(w) = \sqrt{w}$

$f = g \circ h$

$\sqrt{w} = |w|^{1/2} e^{i\frac{\arg w}{2}}$



$z^2 - 1 \notin (-\infty, 0]$

$z^2 \notin (-\infty, 1]$

z not imag., or

$z \notin [-1, 1]$

NOT A CONNECTED DOMAIN. → REJECT

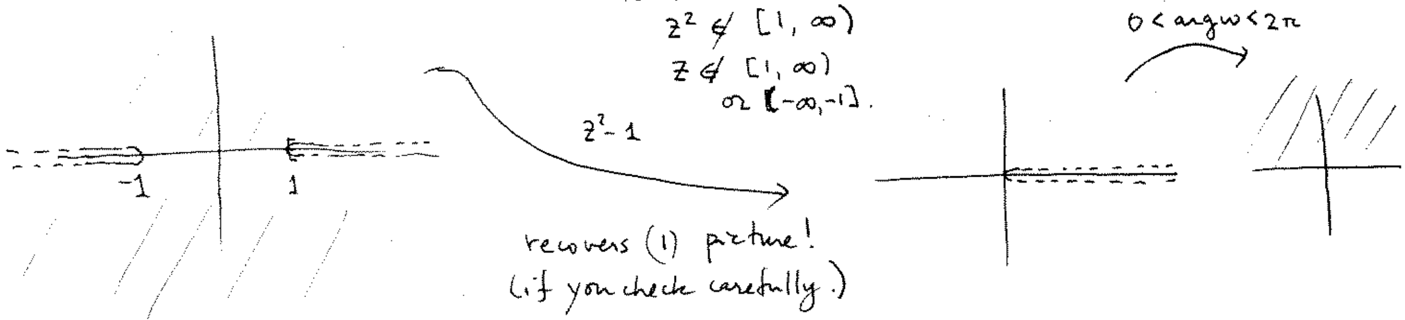
<u>fix</u>: $z^2 - 1 \notin [0, \infty)$ is more restrictive:

$z^2 \notin [1, \infty)$

$z \notin [1, \infty)$ or $[-\infty, -1]$.

$\sqrt{w} = |w|^{1/2} e^{i\frac{\arg w}{2}}$

$0 < \arg w < 2\pi$



recovers (1) picture! (if you check carefully.)