Math 4200
Monday Sept 10

• Discuss connectivity and chain rule for curves
  application from Monday.

Theorem: Let \( A \subseteq \mathbb{C} \) open and connected
\( f: A \to \mathbb{C} \) analytic, with \( f'(z) = 0 \) \( \forall z \)

Then \( f \) is constant

Proof: Let \( z_0 \in A \), \( f(z_0) = c \).
\( B = \{ z \in A \mid f(z) = c \} \)

• \( B \) is closed because \( f \) is continuous and
  \( B = f^{-1}(\{c\}) \) is the preimage of a closed set

• \( B \) is open: \( \) let \( z_1 \in B \)
  since \( A \) is open \( \exists D(z_0;r) \subseteq A \)
  we show \( D(z_0;r) \subseteq B \):
  Proof: let \( \epsilon > 0 \in \mathbb{R} \)
  \( \) let \( \gamma(t) = (1-t)z_1 + t z_2 \) \( \) \( 0 \leq t \leq 1 \)
  be the segment from \( z_1 \) to \( z_2 \),
  contained in \( D(z_0;r) \).

  then \( f(z_2) - f(z_1) = \int_0^1 \frac{d}{dt} f(\gamma(t)) \, dt \)
  (F.T.C., applied to Ref, Imf)

  \[ = \int_0^1 f'(\gamma(t)) \gamma'(t) \, dt \]
  (chain rule for curves)

  \[ = 0 \Rightarrow f(z_2) = f(z_1) = c. \]

• Since \( B \) is open \& closed and non-empty, and since
  \( A \) is connected, \( B = A \)

Def: \( \gamma = [\gamma_1, \gamma_2, \ldots, \gamma_k] \) is a piecewise \( C^1 \) path if each
\( \gamma_i: [a_i, b_i] \to \mathbb{C} \) is \( C^1 \)
and \( \gamma_i(b_i) = \gamma_{i+1}(a_i) \) \( \) \( i = 1, 2, \ldots, k-1 \).

Remark: By reparameterizing one can always assume
\( a_{i+1} = b_i \) \( \) \( i = 1, 2, \ldots, k-1 \),
so that \( \gamma \) corresponds to a continuous path on the single interval
\( [a_1, b_k] \).

Def: \( A \) is pathwise connected iff \( \forall \) \( z_0, z_1 \in A \) \( \exists \) cont. path \( \gamma: [a, b] \to A \)
with \( \gamma(a) = z_0 \)
\( \gamma(b) = z_1 \).
Theorem: If $A \subset C$ is open, then $A$ is connected if and only if $A$ is pathwise connected. Furthermore, in this case, $\forall z_0, z_1, z_2 \in A$ there is a pathwise $C^1$ path $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_k]$ connecting $z_0$ to $z_1$. (Each $\gamma_i$ can in fact be taken as a constant speed line segment parameterization.)

pf: Let $A$ be connected (and open).

Pick $z_0 \in A$.

Let $B \subset A$, $B = \{ z \in A \ s.t. \exists \text{ p.w. C}^1 \text{ path from } z_0 \text{ to } z \}$. 

- $B$ is open: if $z_1 \in B$, then $\exists \gamma \text{ s.t. } D(\gamma, r) \subset A$ for any $w \in D(\gamma, r)$, $\gamma$ line-segment path from $z_1$ to $w$, which can be amalgamated with the p.w. $C^1$ path from $z_0$ to $z_1$ to give a p.w. $C^1$ path from $z_0$ to $w$.

- $B$ is closed: let $z_1 \in \overline{B} \subset A$; $\exists \gamma \text{ s.t. } D(\gamma, r) \subset A$ for any $w \in D(\gamma, r)$.

So $\exists$ p.w. $C^1$ path from $z_0$ to $w$ amalgamating this with a line-segment path from $w$ to $z_1$, so that $z_1 \in B$. Thus $\overline{B} = B$ (in $A$).

Now assume $A$ is path connected.

If $A$ is not connected $\exists B$ s.t. $B \subseteq A$, $B \neq \emptyset$, $B$ is open and closed in $A$.

Pick $z_0 \in B$, $z_1 \in A \setminus B$ and $\gamma : [a, b] \rightarrow A$ a continuous path, $\gamma(a) = z_0$, $\gamma(b) = z_1$.

Let $t_1 = \sup \{ t \in [a, b] \ s.t. \gamma(t) \in B \}$.

If $\gamma(t_1) \notin B$ then $B$ open $\Rightarrow \exists \gamma \text{ s.t. } D(\gamma(t_1), r) \subset A$ $\gamma$ cont. $\Rightarrow \exists \gamma \text{ s.t. } D(\gamma(t_1), r) \subset A$.

Thus $A$ is connected.

This part of the proof does not require $A$ open; $A$ path connected $\Rightarrow A$ connected in general.
Harmonic conjugates

Recall, if \( f(z) = u(x,y) + iv(x,y) \) is analytic and \( C^2 \) (\( u,v \) have continuous partials through 2nd order), then \( u \) (and \( v \)) are harmonic, since

\[
\begin{align*}
\nabla \left\{ \begin{array}{l}
\nabla u = \nabla v \\
\n\nabla v = -\nabla u
\end{array} \right. \Rightarrow u_{xx} = v_{yx} \quad \Rightarrow \quad u_{xx} + u_{yy} = 0
\end{align*}
\]

\( \text{and} \quad v_{yy} = u_{xy} \)

\( v_{xx} = -u_{yx} \)

so \( v_{xx} + v_{yy} = 0 \)

In this case, \( v \) is called the harmonic conjugate of \( u \).

Theorem. If \( u(x,y) \) is harmonic and \( C^2 \) in an open simply connected domain (e.g. \( D(2,0;1) \)), then

\( \exists \) harmonic conjugate \( v(x,y) \), unique up to an additive constant.

\( \text{pf:} \quad u(x,y) \in C^2 \) is given. The system for \( v(x,y) \) is

\[
\begin{align*}
\nabla v &= P(x,y) \quad (= -uy) \\
\n\nabla v &= Q(x,y) \quad (= ux)
\end{align*}
\]

When you study conservative vector fields and Green’s Theorem in multivariable calc, you learn that you can antidifferentiate to find \( v \) iff \( \nabla P = \nabla Q \), which holds since \( P_y = -u_{yy} = u_{xx} = Q_x \) since \( u \) is harmonic.

**Example** \( u(x,y) = xy \)

Show \( u \) is harmonic & find conjugate

\[
\begin{align*}
\nabla v &= \text{Polar form of conjugate} \\
\n\n\text{Integrate CR in polar cords to illustrate why domains which are not simply connected may not have global harmonic conjugates} \\
\end{align*}
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