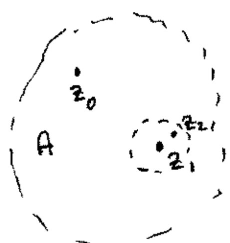
Math 4200
Monday Sept 10

- Discuss connectivity and chain rule for curves application from Friday:

<u>Theorem</u> Let $A \subset \mathbb{C}$ <u>open</u> and <u>connected</u>
$f: A \to \mathbb{C}$ analytic, with $f'(z) = 0 \ \forall z$
Then f is constant

<u>proof</u>: Let $z_0 \in A$, $f(z_0) = c$.
Let $B = \{z \in A \text{ s.t. } f(z) = c\}$

- B is closed because f is continuous and $B = f^{-1}(\{c\})$ is the preimage of a closed set
- B is open: let $z_1 \in B$
  since A is open $\exists\ D(z_1; r) \subset A$
  we show $D(z_1; r) \subset B$:
  pf: Let $|z_2 - z_1| < r$
  Let $\gamma(t) = (1-t)z_1 + t z_2 \quad 0 \le t \le 1$
  be the segment from z_1 to z_2, (contained in $D(z_1; r)$).



then $f(z_2) - f(z_1) = \int_0^1 \frac{d}{dt} f(\gamma(t))\,dt$

(F.T.C., applied to Re f, Im f).

$= \int_0^1 \underbrace{f'(\gamma(t))}_{\substack{\parallel \\ 0}} \gamma'(t)\,dt$ (chain rule for curves)

$= 0 \Rightarrow f(z_2) = f(z_1) = c.$
$\Rightarrow D(z_1; r) \subset B$

- Since B is open & closed and non-empty, and since A is connected, $B = A$ ∎

<u>Def</u>: $\gamma = [\gamma_1, \gamma_2, \ldots \gamma_k]$ is a piecewise C^1 path if each
$\gamma_i : [a_i, b_i] \to \mathbb{C}$ is C^1
and $\gamma_i(b_i) = \gamma_{i+1}(a_i) \quad i = 1, 2, \ldots k-1.$

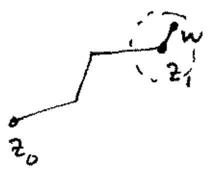
<u>Remark</u>: By reparametrizing one can always assume $a_{i+1} = b_i \quad i = 1, \ldots k-1$, so that γ corresponds to a continuous path on the single interval $[a_1, b_k]$.

<u>Def</u> A is <u>pathwise connected</u> iff $\forall z_0, z_1 \in A\ \exists$ cont. path $\gamma: [a,b] \to A$
with $\gamma(a) = z_0$
$\gamma(b) = z_1$

Theorem If $A \subset \mathbb{C}$ is open then A is connected iff A is pathwise connected. Furthermore in this case, $\forall z_0, z_1 \in A$ $\exists$ a piecewise C' path $\gamma = [\gamma_1, \gamma_2 .. \gamma_k]$ connecting z_0 to z_1.
(Each γ_i can in fact be taken as a constant speed line segment parameterization.)

pf: Let A be connected (and open).
Pick $z_0 \in A$.
Let $B \subset A$, $B := \{ z \in A \text{ s.t. } \exists \text{ p.w. } C' \text{ path from } z_0 \text{ to } z \}$.

- B is open: if $z_1 \in B$, then $\exists r > 0$ s.t. $D(z_1; r) \subset A$
  for any $w \in D(z_1; r)$ $\exists$ line-segment path from z_1 to w, which can be amalgamated with the p.w. C' path from z_0 to z_1 to give a p.w. C' path from z_0 to w.



- B is ~~open~~ closed: Let $z_1 \in \overline{B} \subset A$; $\Rightarrow \exists r > 0$ s.t. $D(z_1; r) \subset A$
  $\Rightarrow \exists w \in B \cap D(z_1; r)$.
  So $\exists$ p.w. C' path from z_0 to w
  amalgamate this with a line-segment path from w to z_1, so that $z_1 \in B$
  Thus $\overline{B} = B$ (in A).

$\Rightarrow B = A$ $\blacksquare$

Now assume A is path connected.
If A is not connected $\exists B$ s.t. $B \subset A$, $B \neq \emptyset, A$, B is open and closed in A.
pick $z_0 \in B$, $z_1 \in A \setminus B$
and $\gamma : [a,b] \to A$ a continuous path, $\gamma(a) = z_0$, $\gamma(b) = z_1$
Let $t_1 = \sup \{ t \in [a,b] \text{ s.t. } \gamma(t) \in B \}$
If $\gamma(t_1) \in B$ then B open $\Rightarrow \exists r > 0$ s.t. $D(\gamma(t_1); r) \subset A$
γ cont $\Rightarrow \exists t > t_1$ s.t.
$\gamma(t) \in D(\gamma(t), r) \subset A$.
$\Rightarrow\!\Leftarrow$.
If $\gamma(t_1) \notin B$ then $A \setminus B$ open
$\Rightarrow \exists r > 0$ s.t. $D(\gamma(t_1); r) \cap B = \emptyset$
$\Rightarrow \not\exists \{t_k\} \nearrow t_1$ s.t. $\gamma(t_k) \in B$. $\Rightarrow\!\Leftarrow$.

Thus A is connected. $\blacksquare$

(this part of the proof does not require A open;
A path connected $\Rightarrow$ A connected in general.)

Harmonic conjugates

Recall, If $f(z) = u(x,y) + i v(x,y)$ is analytic and C^2 (u,v have continuous partials thru 2^{nd} order)

then u (and v) are harmonic,

since

$$CR \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \implies \begin{matrix} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} \end{matrix} \implies u_{xx} + u_{yy} = 0 \qquad \left(\begin{matrix} \text{and } v_{yy} = u_{xy} \\ v_{xx} = -u_{yx} \\ \text{so } v_{xx} + v_{yy} = 0 \end{matrix} \right)$$

n this case, v is called the harmonic conjugate of u

<u>Theorem</u> If $u(x,y)$ is harmonic and C^2 in an <u>open</u> <u>simply connected</u> domain (e.g. $D(z_0;r)$), then $\exists$ harmonic conjugate $v(x,y)$, unique up to an additive constant.

↑ roughly "no holes" we'll make this precise later

<u>pf</u>: $u(x,y) \in C^2$ is given. The system for $v(x,y)$ is

$$\begin{aligned} v_x &= P(x,y) \qquad (= -u_y) \\ v_y &= Q(x,y) \qquad (= u_x) \end{aligned}$$

When you study conservative vector fields and Green's Thm in multivariable calc you learn that you can antidifferentiate to find v iff $P_y = Q_x$, which holds since $P_y = -u_{yy} = u_{xx} = Q_x$ since u is harmonic. ∎

<u>example</u> $u(x,y) = xy$

Show u is harmonic & find conjugate

<u>example</u> : $u(x,y) = \ln\sqrt{x^2+y^2}$ is harmonic in $\underline{\mathbb{C} \setminus \{0\}}$. ($u = Re(\log z)$), or check directly

Integrate CR in polar coords to illustrate why domains which are not simply connected may not have global harmonic conjugates

$$\boxed{\begin{matrix} u_\theta = -r v_r \\ v_\theta = r u_r \end{matrix}}$$