# Math 4200
# December 11, 2002
# FINAL EXAM

Each complete problem below is worth 25 points. Choose any six out of the ten problems to do. Indicate clearly which six you want graded, if you have attempted more. This exam is closed book and closed note. Show complete work for complete credit. Good Luck!

1a) Define what it means for a function f(z) to be complex differentiable (analytic) at a point $z_o$ in C.

(7 points)

1b) State and prove a theorem which relates complex differentiability of f(x+iy)=u(x,y) + iv(x,y) at $z_o=x_o+i\,y_o$ to the Cauchy-Riemann equations and the real differentiability of

$$F(x,y)=\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$

(18 points)

2) Find a branch of

$$f(z)=\sqrt{z^2-z}$$

in a suitable simply connected domain. Make sure to define f precisely.

(25 points)

3a) Find the Laurent series for the function

$$f(z)=\frac{1}{z^2+2\,z}$$

valid in the annulus $0<|z|<2$.

(10 points)

3b) Use your work from part (3a) to compute the contour integral (counterclockwise)

$$\int_{|z|=1} \frac{1}{z^2+2\,z}\,dz$$

(5 points)

3c) Check your work in (3b) using the residue theorem.

(10 points)

4a) Define what it means for a domain to be simply connected

(5 points)

4b) Prove that the complex plane with the origin deleted is not simply connected.

(20 points)

5) Use a contour integral to compute

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+4}\,dx$$

As always, justify all steps.

(25 points)

6) Use a contour integral to find

$$\int_{0}^{2\pi} \cos(\theta)^6\,d\theta$$

(25 points)

7a) Consider the upper half disk consisting of points for which $|z|<2$ and $\mathrm{Re}(z)>0$. Find a conformal map f of this region to the upper half plane so that $f(i)=i$.

(20 points)

7b) Is your map in (7a) unique? Explain.

(5 points)

8a) State precisely what it means for two closed curves $\gamma_0$ and $\gamma_1$ to be homotopic in a domain A.

(8 points)

8b) Find a homotopy between the curves

$$\gamma_0(t)=e^{(it)}+2\,e^{(6it)}$$

and

$$\gamma_1(t)=e^{(6it)}$$

in the complex plane with the origin deleted. (Here we take t between 0 and 2*Pi.) As always, justify!

(7 points)

8c) Use (8b) to find the value of

$$\int_{\gamma_0} 3\frac{1}{z}\,dz$$

(10 points)

9a) Prove that there is no conformal map from the entire complex plane to the unit disk.

(10 points)

9b) Prove that the only conformal bijections for which the entire complex plane is the domain, are the non-constant affine functions f(z)=az+b.

(15 points)

10) State and prove a theorem from this course which you find interesting, and which is important. (Of course, you should choose a theorem which you have not proven elsewhere on this exam.)

(25 points)