Math 4200
Friday Oct 5

HW for Fri Oct 19
2.4 2, 3, 4, 7, 8, 9, 12, 16, 17, 18

Because we reviewed for the Wednesday exam on Monday,
we actually have to go back a week to pick up our mathematical thread....
(so I've recopied the relevant pages)

Finish proving (last Friday notes)

<u>Index Theorem</u>: Let $\gamma : [0,1] \to \mathbb{C} \setminus \{z_0\}$ be a p.w. C^1, closed path.

Then for a unique $n \in \mathbb{Z}$, γ is homotopic to $\gamma^n(t) = z_0 + e^{2\pi i n t}$, $0 \le t \le 1$

and so
$$I(\gamma; z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz = \frac{1}{2\pi i}\int_{\gamma^{(n)}} \frac{1}{z-z_0}\,dz = n$$

"index of γ w.r.t z_0", or "winding number of γ about z_0"

(in practice $I(\gamma; z_0)$ can be computed without reparameterizing γ so that its domain is $[0,1]$)

we've done:

<u>Step 1</u> Translate discussion to origin:

$\gamma_0(t) = \gamma(t) - z_0$ avoids 0, and $\frac{1}{2\pi i}\int_{\gamma_0} \frac{1}{z}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz$

If $\gamma_0(t)$ is homotopic to the path $t \to e^{2\pi i n t}$

then $\gamma(t)$ is homotopic to $t \to z_0 + e^{2\pi i n t} = \gamma^n(t)$

<u>Step 2</u> Let $\gamma_1(t) = \dfrac{\gamma_0(t)}{|\gamma_0(t)|}$ i.e. the projection of γ_0 radially onto the unit circle

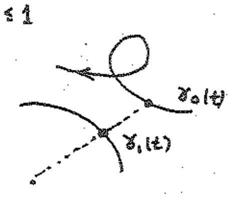
γ_1 is homotopic to γ_0 as closed curves in $\mathbb{C} \setminus \{0\}$ by

$$H(s,t) = (1-s)\gamma_0(t) + s\,\gamma_1(t) \qquad 0 \le s \le 1$$

<u>Step 3</u> Let $\gamma_1(0) = \gamma_1(1) = e^{i\theta}$

Let $\gamma_2(t) = e^{-i\theta}\gamma_1(t)$ (rotate γ_1 by $-\theta$)

Then γ_1 is homotopic to γ_2 as closed curves in $\mathbb{C} \setminus \{0\}$:



$$H(s,t) = e^{-i\theta s}\gamma_1(t) \qquad 0 \le s \le 1.$$

<u>Step 5</u> Being homotopic as closed curves ($\cong$) (or, actually, with the same fixed endpoints) in a domain A is <u>transitive</u>:

$$\text{if } \gamma_0 \cong \gamma_1 \text{ and } \gamma_1 \cong \gamma_2 \text{ then } \gamma_0 \cong \gamma_2$$

$$H_1(s,t) \qquad H_2(s,t) \qquad H(s,t) = \begin{cases} H_1(2s,t) & 0 \le s \le 1/2 \\ H_2(2s-1,t) & 1/2 \le s \le 1 \end{cases}$$

Thus, our

$$\gamma_0 \cong t \to e^{2\pi i n t},$$

$$(\text{so } \gamma \cong \gamma^{(n)})$$

which leaves

<u>Step 4</u> $\gamma_2(0) = \gamma_2(1) = 1$. $\gamma_2(t)$ is moving back & forth around $|z|=1$, as $0 \le t \le 1$.

<u>If</u> we can find a continuous fcn $\theta(t)$ s.t. $\theta(0) = 0$ and

$$\gamma_2(t) = e^{i\theta(t)}$$

then $\theta(t)$ is a choice for $\arg(\gamma_2(t))$, so $\theta(1) = 2\pi n$ some $n \in \mathbb{N}$. since $\gamma_2(1) = 1$.

In this case $\gamma_2(t)$ is homotopic to

$$t \longrightarrow e^{2\pi i n t} \quad (\text{as closed curves on } \mathbb{C} \setminus \{0\})$$

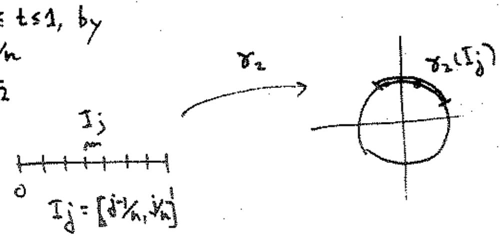
via

$$H(s,t) = e^{i[(1-s)\theta(t) + s(2\pi n t)]} \qquad 0 \le s \le 1.$$

note, $H(s,0) = H(s,1) = 1$

We find $\theta(t)$, continuous on $0 \le t \le 1$, by analysis. Pick n s.t. $|t_1 - t_2| < 1/n$

$$\Rightarrow |\gamma_2(t_1) - \gamma_2(t_2)| < \sqrt{2}$$



This implies that for $I_j = [\frac{j-1}{n}, \frac{j}{n}]$

$\gamma_2(I_j)$ is (connected) an "interval" of the unit circle spanning $< \pi/2$ radians

$\Rightarrow$ if $\arg(\gamma(\frac{j-1}{n}))$ is chosen, i.e. $\theta(\frac{j-1}{n})$

then there is a unique continuous choice of $\theta(t) = \arg(\gamma(t))$

which agrees with this choice for $\theta(\frac{j-1}{n})$

this gives a value for $\theta(j/n)$

in this manner successively define $\theta(t)$ for $I_1, I_2, \ldots I_n$. This yield continuous $\theta(t)$ $0 \le t \le 1$

The definition of index and the deformation theorem leads to the following result, which will underlie almost every other result we prove in this course.
Notice how amazing this result is!

Cauchy Integral formula Let A be open and simply connected
$f: A \to \mathbb{C}$ analytic.
$z_0 \in A$
γ a p.w. C^1 closed contour in A, with z_0 not on the contour

Then
$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)\, I(\gamma; z_0)$$

$f(z_0)$ is determined by the values of $f(z)$ on γ!
(as long as $I(\gamma; z_0) \neq 0$).

proof: Define
$$g(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$$

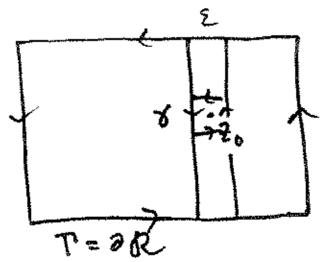
g is analytic in $A \setminus \{z_0\}$, and continuous at z_0.
If we can show the rectangle lemma for g (see below*), then
local antiderivative theorem follows (just used rect. lemma + g continuous)
so, deformation theorem does too.
In particular, g has a global antideriv. G on simply-connected A,
So
$$\int_\gamma g(z)\,dz = 0$$
$$\|$$
$$\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = \int_\gamma \frac{f(z)}{z-z_0}\,dz - f(z_0)\underbrace{\int_\gamma \frac{1}{z-z_0}\,dz}_{2\pi i\, I(\gamma; z_0)}$$

i.e. $$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)\, I(\gamma; z_0) \quad \blacksquare$$

* rectangle lemma for g: let $R = \{x+iy,\ a \le x \le b,\ c \le y \le d\} \subset A$

if $z_0 \notin R$ $\int_{\partial R} g(z)\,dz = 0$ as before.

if $z_0 \in$ interior of R:



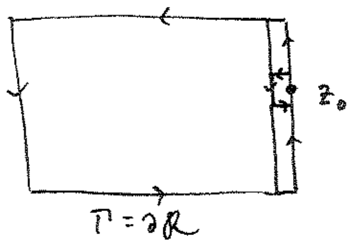
$$\int_\Gamma g(z)\,dz = \int_\gamma g(z)\,dz$$ by rectangle lemmas and contour integral cancellation.

If γ is the bdry of a square with side-lengths ε, then

$$\left|\int_\gamma g(z)\,dz\right| \le \int_\gamma |g(z)|\,|dz| \le 4\varepsilon M$$ where $M = \max_R |g(z)|$.

$$\varepsilon \to 0 \Rightarrow \int_\Gamma g(z)\,dz = 0$$

if $z_0 \in \partial R$,
argument is same as interior case, except use



(or corner picture)