Math 4200
Wed October 31

Where we are:

- uniform limits of analytic functions are analytic (open domain)
- the derivative of the limit is the limit of the derivatives, under uniform convergence.

In particular, if

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

has radius of convergence $R > 0$, then

- $f(z)$ is analytic in $D(z_0; R)$
- $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} = \sum_{j=0}^{\infty} (j+1) a_{j+1} (z-z_0)^j$
- $a_n = \frac{1}{n!} f^{(n)}(z_0)$, so power series are unique, for a given $f(z)$.
- the radius of convergence for $f'(z)$ (which is at least R), actually equals R (it can't go up, because for $|z-z_0| < R$ the series converges absolutely, and $|n a_n| \geq |a_n|$! $(n \geq 1)$).

<u>Exercise 1</u>: Prove $\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$ (which is a more complicated way to see the last bullet point).

by showing $|z-z_0| > R \Rightarrow |a_n (z-z_0)^n| \not\to 0$
$|z-z_0| < R \Rightarrow$ absolute conv. (by ratio test).

<u>Exercise 2</u> $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1 = R$.

So, why does
$$\log(1-z) = -\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$$
for $|z| < 1$? (principal branch of log).

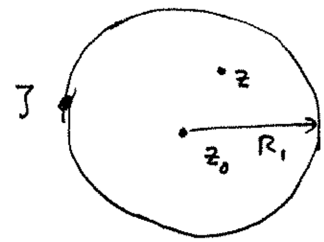
Summary so far: power series give analytic functions.
More surprising: anaytic functions always have power series expansion!

Monday (4)
= Wednesday (2)

Theorem E Let $f(z)$ be analytic in $D(z_0; R)$.
Then, for $a_n = \dfrac{f^{(n)}(z_0)}{n!}$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \forall\ |z-z_0| < R.$$

proof. Let $|z-z_0| = r < R_1 < R$.
Let $\gamma = \{\zeta \mid |\zeta - z_0| = R_1\}$.



C.I.F. $\Rightarrow$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta-z_0|=R_1} \frac{f(\zeta)}{(\zeta-z_0)-(z-z_0)}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta-z_0|=R_1} \frac{f(\zeta)}{\zeta-z_0}\underbrace{\left(\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\right)}_{\substack{\| \text{ page 1!} \\ \sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n}} d\zeta$$

uniform convergence

$$= \frac{1}{2\pi i}\int_{|\zeta-z_0|=R_1} \sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta-z_0}\left(\frac{z-z_0}{\zeta-z_0}\right)^n d\zeta$$

uniform conv.

$$= \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\frac{1}{2\pi i}\int_{|\zeta-z_0|=R_1}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\, d\zeta}_{\frac{1}{n!} f^{(n)}(z_0)}$$

by CIF for derivatives! ■

Interchange integral and limit because integrands converge uniformly!

Exercise 3 Find the power series expansion for $f(z) = \cos z$, centered at $z_0 = 0$. Why, with no additional work, do you know $R = \infty$?

<u>Theorem</u> (multiplying power series term by term). (See HW).

If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$

in $D(z_0;R)$, then

$$f(z)g(z) = a_0b_0 + (a_1b_0 + a_0b_1)(z-z_0) + (a_2b_0 + a_1b_1 + a_0b_2)(z-z_0)^2 + \dots$$
$$= \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} a_j b_{m-j} \right) (z-z_0)^m \quad \text{in } D(z_0;R).$$

slick proof: $f(z)g(z)$ is analytic in $D(z_0;R)$ so has power series.

coeff of $(z-z_0)^m$ is $\frac{1}{m!} (fg)^{(m)}(z_0)$

$(fg)' = f'g + fg'$

$(fg)'' = f''g + 2f'g' + fg'$

induct: $(fg)^{m} = f^{(m)}g + \binom{m}{1} f^{(m-1)}g' + \dots + fg^{(m)}$

@ z_0, $"" = m!\, a_m b_0 + m(m-1)!\, a_{m-1} b_1 + \dots + \frac{m!}{k!} a_{m-k} b_k k! + \dots$ ∎

Amazing "cloning" theorems:

<u>Theorem A</u> : Let A be open and connected (this will be our assumption on domains from now on, unless otherwise stated).

Let $f, g : A \to \mathbb{C}$ analytic.

If $\exists\, D(z_0;r) \subset A$ s.t. $f(z) = g(z)$ $\forall z \in D(z_0;r)$

then $f \equiv g$ on A!

proof: Let $B \subset A$ be defined by

$B = \{ z \in A \text{ s.t. } f^{(n)}(z) = g^{(n)}(z) \ \forall n = 0,1,2,\dots \}$.

$z_0 \in B$ because f and g have the same Taylor series in $D(z_0;r)$

B is closed (in A) because $f^{(n)}(z_k) = g^{(n)}(z_k)$, $z_k \to z \in A \Rightarrow f^{(n)}(z) = g^{(n)}(z)$.

B is open in A because $z_1 \in B$, $D(z_1;r) \subset A \Rightarrow f \equiv g$ in $D(z_1,r)$ (same Taylor series).

$\Rightarrow D(z_1;r) \subset B$.

$\Rightarrow B = A$ ∎

Theorem B $f, g : A \to \mathbb{C}$ analytic.

Let $\{z_k\}_1^\infty \to z_0 \in A$
$\quad\quad \cap$
$\quad\quad A$

If $f(z_k) = g(z_k) \quad \forall k$

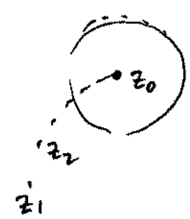
Then $f \equiv g$ in A !

(In particular, taking $g \equiv 0$, all zeroes of an analytic function are isolated, i.e. $g \not\equiv 0$, $g(z_0) = 0 \Rightarrow \exists\, \delta > 0$ s.t. $g(z) \neq 0 \ \forall\ 0 < |z - z_0| < \delta$.)

proof: Consider $h(z) = f(z) - g(z)$.

By hypothesis,

$$0 = h(z_k) = h(z_0)$$
$$k \in \mathbb{N}$$



$z_0 \in A \Rightarrow \exists\, r > 0,\ D(z_0; r) \subset A$

Case I: $h(z) = 0 \quad \forall z \in D(z_0; r)$

then Theorem A $\Rightarrow h \equiv 0$ in A, i.e. $f \equiv g$.

Case II: $h(z) = \displaystyle\sum_{n=N}^{\infty} a_n (z - z_0)^n$ in $D(z_0; r)$, $\quad a_N \neq 0$

$$\Rightarrow h(z) = (z - z_0)^N \left(a_N + \sum_{n=N+1}^{\infty} a_n (z - z_0)^{n-N} \right)$$

$$\frac{h(z)}{(z - z_0)^N} = a_N + k(z) \ ; \quad \lim_{z \to z_0} k(z) = 0$$

$$\Rightarrow \lim_{z \to z_0} \frac{h(z)}{(z - z_0)^N} = a_N \neq 0$$

$$\Rightarrow \lim_{z_k \to z_0} \frac{h(z_k)}{(z_k - z_0)^N} \neq 0 \quad \Rightarrow\!\Leftarrow \ !!$$

Thus case I holds.

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