Math 4200
Monday Oct 29

§3.2 : Power series and Taylor's Thm.
Recall the convergence results from Friday

Theorem: Let $f_n : A \to \mathbb{C}$ analytic $n = 1, 2, \cdots$
$A \subset \mathbb{C}$, A open.

Let $\{f_n\} \to f$ on A, uniformly on each compact $K \subset A$.
Then f is analytic, and each derivative sequence
$$f_n^{(k)}(z) \to f^{(k)}(z), \quad \text{uniformly on compact subsets.}$$

In particular,
if $f_n(z) = \sum_{j=0}^{n} g_j(z)$

$$\left.\begin{array}{l} |g_j(z)| \le M_j \quad \forall z \in A \\ \text{and } \sum_{j=0}^{\infty} M_j < \infty \end{array}\right\} \text{"Weierstrasse M test"}$$

then $\sum_{j=0}^{\infty} |g_j(z)|$ is uniformly convergent on A (series is "uniformly absolutely convergent")

and $f_n \to f$ uniformly on A, and f is analytic.

Furthermore, $f_n^{(k)}(z) = \sum_{j=0}^{n} g_j^{(k)}(z) \longrightarrow f^{(k)}(z)$, uniformly on compact sets.

Exercise 1

1a) Show $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots = \frac{1}{1-z}$ for $|z| < 1$. (This is our favorite power series.)

1b) Find a power series for $\frac{1}{(1-z)^2}$, valid for $|z| < 1$.

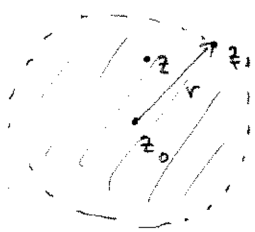
Power series:

Let $\{a_n\}_{n=0,1,2,\ldots}$ be a sequence of complex numbers.

The infinite sum
$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$
is called a power series in z, centered at z_0.

<u>Lemma</u>: If $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges, then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges $\forall z$ s.t. $|z-z_0| < |z_1 - z_0| := r$

In fact, the series converges uniformly absolutely for $|z-z_0| \leq r - \varepsilon$ (any fixed $\varepsilon > 0$).



proof: convergence of series $\Rightarrow$ individual terms approach 0

i.e. $\lim_{n\to\infty} |a_n (z_1 - z_0)^n| = 0$

so $\exists\, M$ s.t. $|a_n| r^n \leq M \quad \forall n \qquad r = |z_1 - z_0|$

Let $|z-z_0| \leq r - \varepsilon$ for some $\varepsilon > 0$.

Then
$$\sum_{n=0}^{\infty} |a_n (z-z_0)^n| = \sum_{n=0}^{\infty} \left| a_n \left[(z_1 - z_0)\left(\frac{z-z_0}{z_1-z_0}\right)\right]^n \right|$$
$$\leq \sum_{n=0}^{\infty} M \left(\frac{r-\varepsilon}{r}\right)^n = \sum_{n=0}^{\infty} M \mu^n \qquad \mu = 1 - \varepsilon/r < 1$$
$$= \frac{M}{1-\mu} < \infty. \quad \blacksquare$$

<u>Theorem B</u> For each power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

$\exists!\ R \in [0, \infty]$, called the <u>radius of convergence</u> s.t.

(i) series converges $\forall z$ s.t. $|z-z_0| < R$, uniformly absolutely for $|z-z_0| \leq R_1 < R$.

(ii) series diverges $\forall z$ s.t. $|z-z_0| > R$

In particular, $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is analytic for $|z-z_0| < R$, and may be differentiated term by term to compute $f'(z)$ or $f^{(k)}(z)$.

<u>Exercise</u> 2a) What is the radius of convergence for the series in Exercise 1?

2b) Find R for $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$

2c) Find R for $\sum_{n=0}^{\infty} n!\, z^n$

Proof of Theorem B: R is unique if it exists, since at most one R can satisfy (i) & (ii).

We claim $R := \sup\{r \geq 0 \text{ s.t. } \sum a_n (z-z_0)^n \text{ conv. } \forall z \text{ s.t. } |z-z_0| \leq r\}$ satisfies (i) & (ii).

(i) If $|z-z_0| \leq R_1 < R$ apply Lemma, with $|z-z_0| \leq R_1 < R_2 < R$. ∎

(ii) If $|z-z_0| > R$ and series converges, then Lemma would imply (after picking $R < R_1 < |z-z_0|$) that series converges for $|z-z_0| \leq R_1$ $\Rightarrow\!\Leftarrow$ ∎

In practice, usually find R using ratio test.
There actually is formula though:

Theorem C $\left(\frac{1}{R}\right) = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ $\left(= \lim_{n\to\infty} \left(\sup_{j \geq n} \sqrt[j]{|a_j|} \right) \right)$

Theorem D If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges for $|z-z_0| < R$, then each a_n is unique

pf $f^{(k)}(z) = \sum_{n=k}^{\infty} a_n (n)(n-1)\cdots(n-k+1)(z-z_0)^{n-k}$

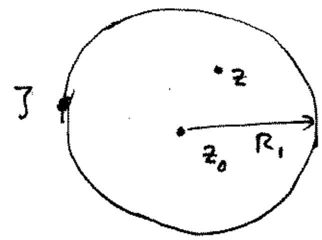
$f^{(k)}(z_0) = k!\, a_k \Rightarrow a_k = \frac{1}{k!} f^{(k)}(z_0)$

∎

<u>Theorem E</u> Let $f(z)$ be analytic in $D(z_0; R)$.
Then, for $a_n = \dfrac{f^{(n)}(z_0)}{n!}$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \forall\ |z-z_0| < R.$$

<u>proof.</u> Let $|z-z_0| = r < R_1 < R$.
Let $\gamma = \{\zeta \mid |\zeta - z_0| = R_1\}$.



C.I.F. $\Rightarrow$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta - z_0| = R_1} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta - z_0| = R_1} \frac{f(\zeta)}{\zeta - z_0}\left(\underbrace{\frac{1}{1 - \frac{z-z_0}{\zeta - z_0}}}_{\substack{\| \text{ page 1!} \\ \sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n}}\right) d\zeta$$

uniform convergence

$$= \frac{1}{2\pi i}\int_{|\zeta - z_0| = R_1} \sum_{n=0}^{\infty} \frac{f(\zeta)}{\zeta - z_0}\left(\frac{z-z_0}{\zeta-z_0}\right)^n d\zeta$$

uniform conv.

$$= \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\frac{1}{2\pi i}\int_{|\zeta - z_0| = R_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta}_{\frac{1}{n!} f^{(n)}(z_0)}$$

interchange integral and limit because integrands converge uniformly!

by CIF for derivatives! $\blacksquare$

<u>Exercise 3</u> Find the power series expansion for $f(z) = \cos z$, centered at $z_0 = 0$. Why, with no additional work, do you know $R = \infty$?