Math 4200
October 26 Friday

Homework for Friday Nov. 2
3.1 4, 6, 7, 14
3.2 2, 3, 4, 7, 11, 13, 14, 18, 20

§3.1 Sequences and series of analytic fns.

Recall a key analysis theorem:

Theorem 1:

If $F_n : A \to \mathbb{R}^\ell$ is cont, $n = 1, 2, \ldots$
$A \subset \mathbb{R}^k$

and if $\{F_n\} \to F$ uniformly $\forall x \in A$ i.e. $\forall \varepsilon > 0 \ \exists N$ s.t. $n \geq N \Rightarrow \|F_n(x) - F(x)\| < \varepsilon$ $\underline{\forall x \in A}$

then F is continuous

proof: Let $x_0 \in A$, let $\varepsilon > 0$.

Pick N s.t. $\|F_N(x) - F(x)\| < \varepsilon/3 \quad \forall x \in A$. (unif. conv.)

Pick δ s.t. $\|x - x_0\| < \delta \Rightarrow \|F_N(x) - F_N(x_0)\| < \varepsilon/3$. ($F_N$ cont)

Then $\|x - x_0\| < \delta \Rightarrow \|F(x) - F(x_0)\| \leq \|F(x) - F_N(x)\| + \|F_N(x) - F_N(x_0)\| + \|F_N(x_0) - F(x_0)\|$
$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$

■

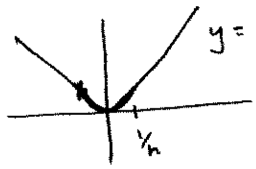
Exercise 1 Give an example of

$\{F_N\}$, $F_N : [0,1] \to \mathbb{R}$, $F_N(x) \to F(x) \quad \forall x \in [0,1]$ (but not uniformly)
F_N continuous

but F is NOT continuous

Also recall $\{F_n\} \to F$ uniformly, $F_n \in C'(A)$ does NOT imply $F \in C'(A)$, or that even if $F \in C'(A)$, $F_n' \to F'$!

e.g.



$$y = F_n(x) = \begin{cases} |x|, & |x| \geq 1/n \\ n x^2 & |x| \leq 1/n \end{cases}$$

$F_n(x) \to |x|$ uniformly.

Life is so much better with analytic functions:

Theorem 2. Let $f_n : \underset{\mathbb{C}}{\overset{}{A}} \to \mathbb{C}$ be analytic, on $A \subset \mathbb{C}$ open, $n = 1, 2, \ldots$

Let $f_n(z) \to f(z)$ $\forall z \in A$, uniformly on each compact subset $K \subset A$.

Then

(1) f is continuous

and

(2) f is analytic.

Furthermore,

(3) the k^{th} derivatives $f_n^{(k)}(z) \to f^{(k)}(z)$, uniformly on each compact subset $K \subset A$.

proof: (1) Let $z_0 \in A$, $\overline{D(z_0;r)} \subset A$.

Since $\{f_n\} \to f$ uniformly on $D(z_0;r) \subset \overline{D(z_0;r)}$,

f is continuous at z_0, by Theorem 1. ∎

(2) Let $z_0 \in A$, $\overline{D(z_0;r)} \subset A$.

Recall Morera's Theorem, which followed from the circle of ideas around the local antidiff thm, combined with CIF for derivatives

→ If f is continuous and satisfies $\oint_\gamma f(z)\,dz = 0$ for each coord rectangle in $D(z_0;r)$, then f is analytic in $D(z_0;r)$.

But $\{f_n\} \to f$ unif. on $D(z_0,r)$ implies

$$0 = \oint_\gamma f_n(z)\,dz \longrightarrow \oint_\gamma f(z)\,dz,$$ since (as we've seen before),

$$\left| \int_\gamma f(z) - f_n(z)\,dz \right| \leq \left[\max\{|f(z) - f_n(z)| \text{ s.t. } z \in \gamma\}\right] L$$

where L = length of γ.

Thus f is analytic by Morera. ∎

(3) If $K \subset A$ is compact then K is covered by a finite union $\bigcup_{j=1}^{N} D(z_j;r_j)$ where $z_j \in K$ and $\overline{D(z_j;2r_j)} \subset A$.

Thus $f_n \to f$ uniformly on each $\overline{D(z_j;2r_j)}$. We use this fact and C.I.F. for derivatives to show $f_n^{(k)} \to f^{(k)}$ uniformly on each $D(z_j,r_j)$, hence uniformly on K:

For $z \in D(z_j,r_j)$

$$f_n^{(k)}(z) = \frac{k!}{2\pi i}\int_{|\zeta - z_j| = 2r_j} \frac{f_n(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta \xrightarrow{n\to\infty} \frac{k!}{2\pi i}\int_{|\zeta - z_j| = 2r_j} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta .$$

because

$$\max\left\{ \left| \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^{k+1}} \right| \text{ s.t. } |\zeta - z_j| = 2r_j \right\} \to 0 \text{ as } n \to \infty$$

$$\wedge\!\!\mid$$

$$\frac{|f_n(\zeta) - f(\zeta)|}{r^{k+1}}$$

∎

Recall there is a 1-1 correspondence between sequences $\{f_n(z)\}$ and series $\sum_{j=1}^{n} g_j(z) = f_n(z) = S_n(z)$; the partial sum

$$g_j(z) = \begin{cases} f_1(z) & j = 1 \\ f_j(z) - f_{j-1}(z) & j > 1 \end{cases}$$

$$f_n(z) = \sum_{j=1}^{n} g_j(z)$$

M for modulus?

<u>Weierstrass M test for uniform (and absolute) convergence</u> of series

Let $\{g_n\}$, $g_n : A \to \mathbb{C}$. (A open as always)
$\widehat{\mathbb{C}}$

If $\exists\, M_n$ s.t. $|g_n(z)| \le M_n \;\; \forall z \in A$
and s.t. $\sum_{n=1}^{\infty} M_n < \infty$, then $S_n(z) = \sum_{j=1}^{n} g_j(z)$ converges uniformly to a limit fcn. on A

In fact, the moduli partial sums $\sum_{j=1}^{n} |g_j(z)|$ converge to a function $\le \sum_{j=1}^{\infty} M_j$ as well.

(This is called <u>absolute convergence</u> of the series)

<u>proof</u>: Let $k < \ell$. Then

$$|S_\ell - S_k| = \left|\sum_{j=k+1}^{\ell} g_j(z)\right|$$

$$\le \sum_{j=k+1}^{\ell} |g_j(z)| \le \sum_{j=k+1}^{\ell} M_j \to 0 \text{ as } k,\ell \to \infty, \text{ since } \sum_{j=1}^{\infty} M_j < \infty.$$

Thus $\{S_k(z)\}$ is uniformly Cauchy. Since $\mathbb{C}$ is complete, $S_k(z) \to S(z)$, uniformly (and uniformly absolutely) in A ∎

Example: Define the usual branch of $\log z = \ln|z| + i \arg z \quad -\pi < \arg z < \pi$, so that $\log n = \ln n, \ n \in \mathbb{Z}$.

then $n^{-z} = e^{-z \log n} = e^{-z \ln n}$ is analytic $\forall z$.

Consider the series

$$\zeta(z) := \sum_{n=1}^{\infty} n^{-z}.$$

If $z = x+iy$ then

$$n^{-z} = e^{-(x+iy)\ln n}$$

$$|n^{-z}| = e^{-x \ln n} = n^{-x}$$

$\sum_{n=1}^{\infty} n^{-x}$ converges for $x > 1$ (uniformly for $x \geq 1+\delta$) by e.g. integral comparison test.

In ODE's or Calculus you may have computed the magic formulas

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$\zeta(z)$ is called the Riemann Zeta function.

It follows from today's theorems that $\zeta(z)$ is defined for $\operatorname{Re} z > 1$ and is analytic.

In fact $\zeta(z)$ extends to $\mathbb{C} \setminus \{1\}$ as an analytic function. The <u>Riemann hypothesis</u> is that all its zeroes lie on the line $\operatorname{Re}(z) = \frac{1}{2}$. (Except for the "trivial zeros" at $z = -2, -4, -6 \ldots$)

The Zeta function has amazing connections to number theory. (as does the <u>Riemann hypothesis</u>).

perhaps to "largest" open problem in mathematics... one of the seven "Clay millenium prize" problems.

Image:Complex zeta.jpg

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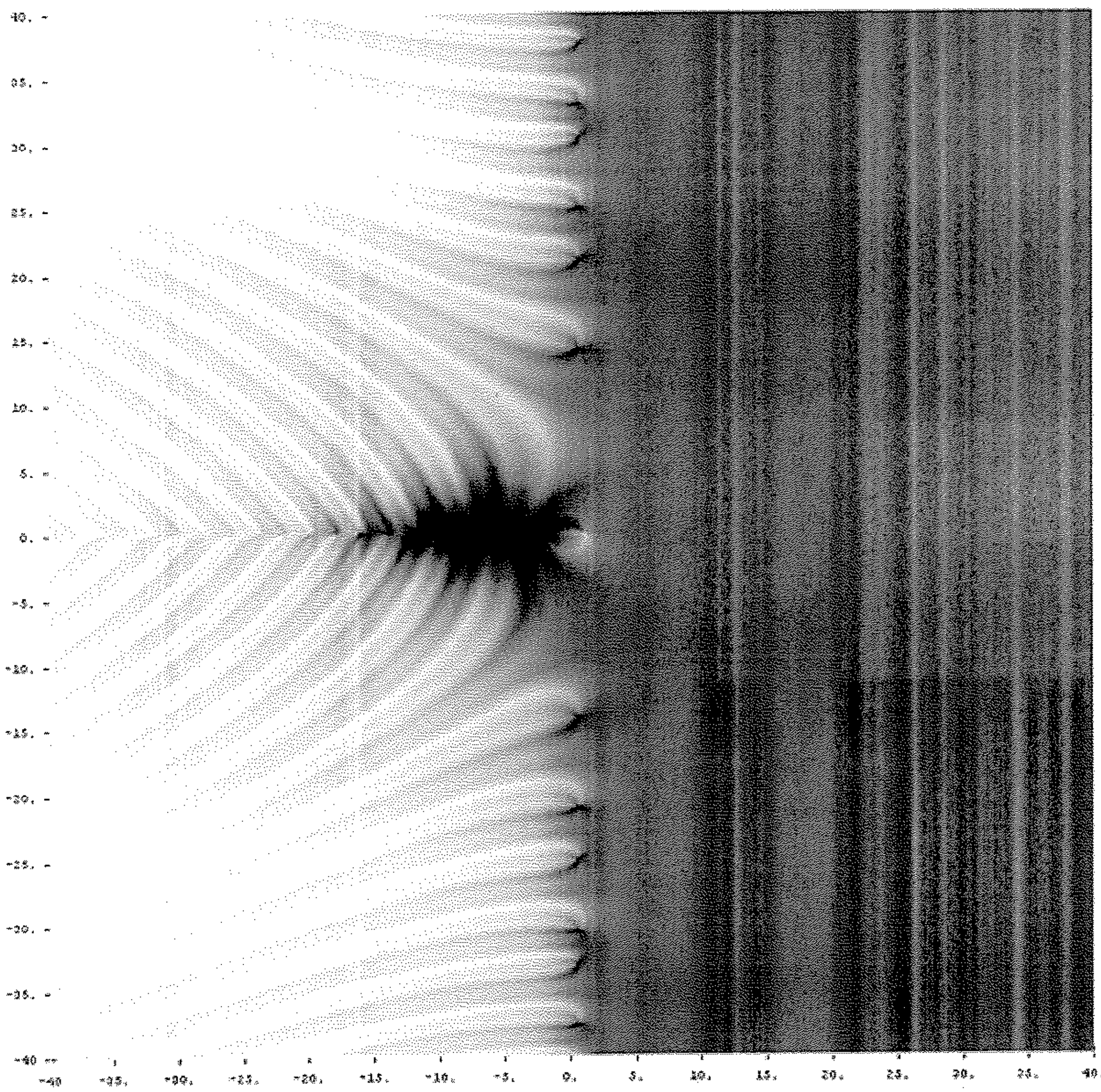

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Beschreibung

colors represent complex value of $\zeta(z)$