Math 4200
Wednesday Oct 24

- Poisson Integral formula for harmonic functions on a disk (page 3 Monday).

Also, there's a clever proof of the existence of a conjugate fcn on simply connected domains A, which uses the antiderivative theorem.
(To get the MVP for harmonic we assumed $u = \operatorname{Re} f$, f analytic. We also make this assumption for the Poisson Integral formula.).

<u>Theorem</u>: Let $A \subset \mathbb{C}$ be simply connected and open.
Let $u \in C^2(A)$ satisfy $u_{xx} + u_{yy} \equiv 0$.
Then $u = \operatorname{Re}(f)$, where f is analytic on A. (So $v := \operatorname{Im}(f)$ is a conjugate to u.)

<u>Pf</u>: Define $g(z) = u_x - i u_y$

CR for g: $(u_x)_x = (-u_y)_y$ ✓
$(u_x)_y = -(-u_y)_x$ ✓

so g is analytic.
so g has an antiderivative, call it $f(z) = U + iV$
(since A is simply connected)

Then $f' = U_x + iV_x = -i(U_y + iV_y)$ CR for f
but $f' = g = u_x - iu_y$ by construction.

Thus $u_x = U_x$, $u_y = U_y$, so, after addition of a const if necessary,
$u = U$.

■

Math 4200

Maple play, October 24

Work for the Poisson integral formula for harmonic functions on the unit disk: This work can be used to show that given any piecewise continuous boundary value function $f(e^{(i\theta)})$, The Poisson integral formula defines a harmonic function with piecewise continuous boundary values given by f. The idea is that the PIF defines an integral superposition of harmonic functions (finite linear combinations of harmonic functions are clearly harmonic, a similar result holds for integral superposition).

```
> with(plots):
> g:=(x,y)->(1-x^2-y^2)/(1-2*x*cos(theta)-2*y*sin(theta)
  +x^2+y^2);
```

$$g := (x, y) \to \frac{1-x^2-y^2}{1-2\,x\cos(\theta)-2\,y\sin(\theta)+x^2+y^2}$$

$\rho e^{i\phi} = x+iy$ in PIF

```
> diff(g(x,y),x,x)+diff(g(x,y),y,y);
```

$$-\frac{4}{1-2\,x\cos(\theta)-2\,y\sin(\theta)+x^2+y^2}+\frac{4\,x\,(-2\cos(\theta)+2\,x)}{(1-2\,x\cos(\theta)-2\,y\sin(\theta)+x^2+y^2)^2}$$
$$+\frac{2\,(1-x^2-y^2)(-2\cos(\theta)+2\,x)^2}{(1-2\,x\cos(\theta)-2\,y\sin(\theta)+x^2+y^2)^3}-\frac{4\,(1-x^2-y^2)}{(1-2\,x\cos(\theta)-2\,y\sin(\theta)+x^2+y^2)^2}$$
$$+\frac{4\,y\,(-2\sin(\theta)+2\,y)}{(1-2\,x\cos(\theta)-2\,y\sin(\theta)+x^2+y^2)^2}+\frac{2\,(1-x^2-y^2)(-2\sin(\theta)+2\,y)^2}{(1-2\,x\cos(\theta)-2\,y\sin(\theta)+x^2+y^2)^3}$$

```
> simplify(%); #gettin zero here shows the sense in which
  #the Poisson integral formula represents
  #a harmonic function as an integral superposition
  #of harmonic functions.
```

$$0$$

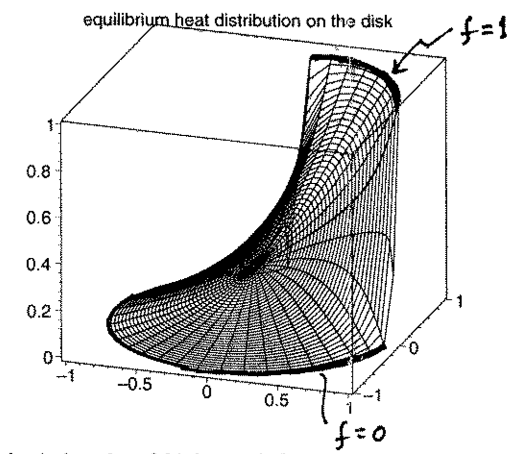
g(x,y) is harmonic!

Here's an example where the boundary values $f(e^{(i\theta)})$ are 1 on a quarter circle $0 < \theta < \frac{\pi}{2}$, and zero on the rest of the circle: (Could you write this function using the imaginary part of some complex logarithm expression?)

```
> u:=(rho,phi)->1/(2*Pi)*evalf(int((1-rho^2)/(1-2*rho*cos(theta-phi)
  +rho^2),theta=0..Pi/2));
```

$$u := (\rho, \phi) \to \frac{1}{2}\,\frac{\mathrm{evalf}\left(\int_0^{\frac{\pi}{2}} \frac{1-\rho^2}{1-2\,\rho\cos(\theta-\phi)+\rho^2}\,d\theta\right)}{\pi}$$

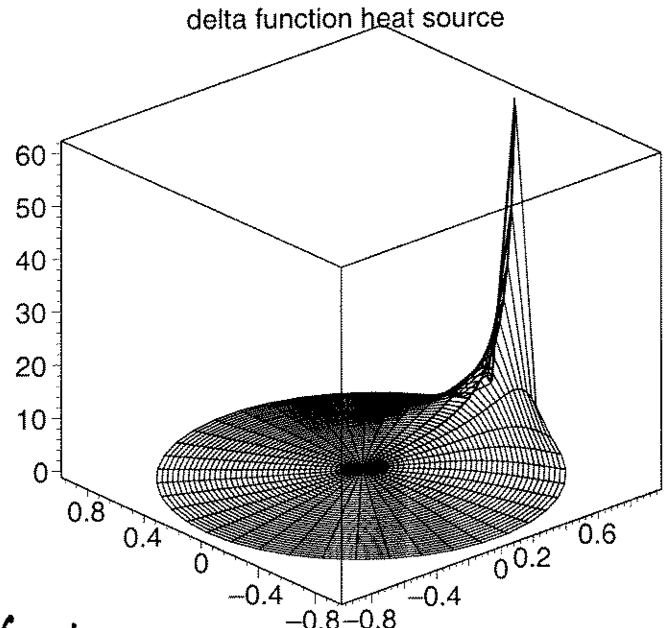
```
> plot3d([rho*cos(phi),rho*sin(phi),u(rho,phi)],rho=0.01..(.99),phi=
  0..2.05*Pi,grid=[40,40],axes=boxed,style=wireframe,
  color=black,title='equilibrium heat distribution on the disk');
```



So, without any work, what is the value of this harmonic function at the origin - precisely?

Understanding the Poison integral as an integral superposition of source functions:

```
> plot3d([rho*cos(phi),rho*sin(phi),(1/2*Pi)*
  (1-rho^2)/(1-2*rho*cos(phi)+rho^2)],rho=0..(.95),
  phi=0..2.05*Pi,grid=[40,40],axes=boxed,style=wireframe,
  color=black,title=`delta function heat source`);
```



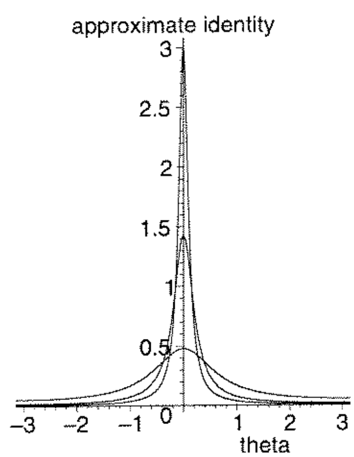
delta function

Illustrating the "approximate identity" nature of the function g:

```
> g:=(rho,theta)->1/(2*Pi)*(1-rho^2)/(1-2*rho*cos(theta)+rho^2);
```

$$g := (\rho, \theta) \to \frac{1}{2} \frac{1-\rho^2}{\pi\,(1-2\,\rho\cos(\theta)+\rho^2)}$$

```
> plot({g(.5,theta),g(.8,theta),g(.9,theta)},theta=-Pi..Pi,
  color=black,title=`approximate identity`);
```



tarmonic functions describe equilibrium heat (temperature) distributions.
They are also potential functions in electrostatics and fluid mechanics.
Using the PIF. and conformal transformations one can find important harmonic functions in other domains besides the disk.
This circle ~~of~~ of ideas contains several interesting project directions.