Math 4200

Oct 22 Monday. (details)

We're in the middle of proving the maximum modulus principle, via the mean value property.

<u>Theorem</u>: $A \subset \mathbb{C}$, A open, connected, bounded

$f: A \to \mathbb{C}$ analytic, $f: \bar{A} \to \mathbb{C}$ continuous.

Let $M = \max\{|f(z)| \text{ s.t. } z \in \bar{A}\}$.

Then $M = \max\{|f(z)| \text{ s.t. } z \in \partial A\}$ (max occurs on the bdry).

Furthermore, if $\exists\, z_0 \in A$ s.t. $|f(z_0)| = M$ then f is constant, $f(z) \equiv f(z_0)$

<u>pf</u>: It suffices to show that if $\exists\, z_0 \in A$ s.t. $|f(z_0)| = M$, then f is constant.

$$\left.\begin{array}{l}\text{Let } B = \{z \in A \text{ s.t. } |f(z)| = M\}. \text{ We first show } B \text{ is open and closed in } A, \\ \text{which implies } B = A\end{array}\right\} \text{Step 1}$$

$$\left.\begin{array}{l}\text{Then we show that} \\ |f(z)| \equiv M \Rightarrow f(z) \equiv f(z_0)\end{array}\right\} \text{Step 2}$$

<u>Step 1</u>

<u>pf</u>: B is not empty, since $z_0 \in B$.

B is closed: (in A): Let $\{z_k\} \subset B$, $z_k \to z \in A$. Then $|f(z_k)| = M$, $|\ |\circ f$ cont $\Rightarrow |f(z)| = M$ $\Rightarrow z \in B$. ■

B is open: Let $z_1 \in B$, $D(z_1; R) \subset A$. We show $D(z_1; R) \subset B$:

pf: Let $0 < r < R$.

$$\text{MVP} \Rightarrow f(z_1) = \frac{1}{2\pi}\int_0^{2\pi} f(z_1 + re^{i\theta})\,d\theta$$

$$\boxed{\Rightarrow M = |f(z)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_1 + re^{i\theta})|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} M\,d\theta = M}$$

If $|f(z_1+re^{i\theta})|$ is <u>NOT</u> $\equiv M$ $(0 \le \theta \le 2\pi)$ then this is a strict inequality which would be a contradiction.

Thus $|f(z_1 + re^{i\theta})| = M$

$\forall\, 0 < r < R,\ 0 \le \theta \le 2\pi$

■

This follows from the real analysis

<u>Lemma</u>: Let $g(t) \le M$ on $[a,b]$, $g \in C([a,b])$.

Then $\int_a^b g(t)\,dt \le M(b-a)$ with equality iff $g(t) \equiv M$, $a \le t \le b$.

<u>Step 2</u> $|f(z)| \equiv M \Rightarrow f(z) \equiv f(z_0)$

pf: (better than before).

Write $f = u + iv$.

Then $|f|^2 = M^2 = u^2 + v^2$. If $M = 0$ then $f(z) \equiv 0$ & done.

Else,
$$2(uu_x + vv_x) \equiv 0$$
$$2(uu_y + vv_y) \equiv 0$$

$$\Rightarrow \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\ne 0} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \text{CR} \Rightarrow 0 = \det \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} = u_x^2 + v_x^2 \ (= u_y^2 + v_y^2) = |f'(z)|^2.$$

Thus matrix is singular!

Thus $f' \equiv 0$ so $f \equiv f(z_0)$.

Fri ③
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Exercise 1 : Suppose two analytic functions on A are continuous on $\bar{A}$, and have the same values on ∂A.
What can you say about the two functions? (A is bounded, open, connected)

Sol'n: Let f & g be these two functions.

Then $h = f - g$ is analytic. $h = 0$ on $\partial A \Rightarrow |h| = 0$ on $\partial A \Rightarrow |h| \equiv 0$ in A by maximum modulus principle.

Thus $h \equiv 0$, i.e. $f \equiv g$!

Exercise 2 If u is harmonic (and C^2) on A and continuous on $\bar{A}$, (A is bd, open connected)
what can you say about $M = \max\{u(z),\ z \in \bar{A}\}$
$m = \min\{u(z),\ z \in \bar{A}\}$?

Ans: M & m are attained on the boundary. If either is attained as ~~...~~ $u(z_0)$ with $z_0 \in A$, then u is constant!

proof: in case $\exists\, z_0 \in A$ with $u(z_0) = M$, show $B := \{z \in A \text{ s.t. } u(z) = M\}$ is open and closed in A.

B closed in A ✓

B open in A: Let $z_1 \in B$, $D(z_1; R) \subset A$.

Write analogous ineq to page 1: Let $0 < r < R$

$$M = u(z_1) = \frac{1}{2\pi}\int_0^{2\pi} u(z_1 + re^{i\theta})\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} M\, d\theta = M$$

strict ineq unless $u(z_1 + re^{i\theta}) = M$ $\forall\, 0 \le \theta \le 2\pi$. ■

in case $\exists\, z_0 \in A$ with $u(z_0) = m$, proof is same, except $\ge$ replaces $\le$

Exercise 3. The Dirichlet Problem for harmonic functions is:

Let $A \subset \mathbb{C}$, ∂A piecewise C^1. Let $f : \partial A \to \mathbb{R}$ be continuous (the boundary values).
Find $u \in C^2(A) \cap C(\bar{A})$ s.t.

$$\begin{cases} \Delta u = 0 & \text{in } A \\ u = f & \text{on } \partial A. \end{cases}$$

Why can there be at most 1 solution to the Dirichlet Problem?

ans Because if u, v are 2 sol'ns then $w = u - v$ is also harmonic and is zero on the boundary. Thus $m = 0 = M$ so $w \equiv 0$; so $u \equiv v$ ■

There's an analogous formula to the Cauchy integral formula for harmonic functions. In the case the domain is a disk it's called the Poisson Integral formula:

Let u be harmonic on $D(0;r)$, continuous on $\overline{D(0;r)}$. Then for $\rho<r$,

$$u(\rho e^{i\phi}) = \frac{r^2-\rho^2}{2\pi}\int_0^{2\pi} \frac{u(re^{i\theta})}{r^2-2r\rho\cos(\phi-\theta)+\rho^2}\,d\theta$$

← text typo

pf: We'd like to just take the real part of C.I.F.

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)}{\zeta-z}\,d\zeta \quad , \ |z|<r$$

$$= \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(re^{it})}{re^{it}-z}\,\cancel{i}re^{it}\,dt$$

In fact, if $u(re^{i\theta}) := f(re^{i\theta})$ $0\le\theta\le 2\pi$, this formula gives a (unique) soln to Dirichlet Problem, provided f is continuous

but, unfortunately, u & v get jumbled on RHS.

A mysterious ("reflection point") trick saves us.

For $|z|<r$ define $\tilde{z} = \frac{r^2}{|z|^2}z$ $\left(=\frac{r^2}{\bar{z}}\right)$, the "reflection" of z through the circle $|\zeta|=r$

then

$$f(z) = \frac{1}{2\pi i}\int_{|z|=r} f(\zeta)\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\tilde{z}}\right)d\zeta \qquad \text{because } I(\gamma;\tilde{z})=0\,!$$

inscrutible calculation: $|\zeta|=r$

$$\frac{1}{\zeta-z}-\frac{1}{\zeta-\tilde{z}} = \frac{1}{\zeta-z}-\frac{1}{\zeta-\frac{r^2}{\bar{z}}}$$

$$= \frac{1}{\zeta-z}-\frac{\bar{z}}{\zeta\bar{z}-r^2}$$

$$= \frac{1}{\zeta-z}-\frac{\bar{z}}{\zeta(\bar{z}-\bar{\zeta})}$$

$$= \frac{1}{\zeta-z}+\frac{\bar{z}}{\zeta(\bar{\zeta}-\bar{z})}$$

$$= \frac{\zeta(\bar{\zeta}-\bar{z})+\bar{z}(\zeta-z)}{\zeta|\zeta-z|^2}$$

$$= \frac{|\zeta|^2-|z|^2}{\zeta|\zeta-z|^2}$$

so, using calc. at left (and writing $z=\rho e^{i\phi}$)

$$(u+iv)(z) = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} f(re^{i\theta})\left(\frac{r^2-|z|^2}{|re^{i\theta}-\rho e^{i\phi}|^2}\right)\frac{1}{\cancel{re^{i\theta}}}\,\cancel{ire^{i\theta}}\,d\theta$$

Now, take real part:

$$u(\rho e^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-\rho^2)\,u(re^{i\theta})}{(r\cos\theta-\rho\cos\phi)^2+(r\sin\theta-\rho\sin\phi)^2}\,d\theta$$

gives *
assuming u harmonic on $\underset{\text{open}}{A}$, $\overline{D(0;r)}\subset A$,
since we can construct conjugate v.
Limiting argument gives P.I.F.