We're in the middle of proving the maximum modulus principle, via the mean value property.

**Theorem:** \( A \subseteq \mathbb{C} \), \( A \) open, connected, bounded
\[ f: A \to \mathbb{C} \text{ analytic, } f: \overline{A} \to \mathbb{C} \text{ continuous.} \]

Let \( M = \max \{ |f(z)| : z \in A \} \).

Then \( M = \max \{ |f(z)| : z \in \partial A \} \) (max occurs on the boundary).

Furthermore, if \( f(z_0) = M \) then \( f \) is constant: \( f(z) \equiv f(z_0) \)

**pf:** It suffices to show that if \( \exists z_0 \in A \text{ s.t. } f(z_0) = M \), then \( f \) is constant.

Let \( B = \{ z \in A \text{ s.t. } |f(z)| = M \} \).

**Steps:**
1. We show \( B \) is open and closed in \( \overline{A} \),
   which implies \( \overline{B} = A \).
2. Then we show that
   \[ |f(z_0)| = M \implies f(z) \equiv f(z_0). \]

**Step 1**
\( B \) is not empty, since \( z_0 \in B \).

\( B \) is closed:
1. \( \text{(in } A) \): Let \( \{ z_n \} \subseteq B, z_n \to z \in A \).
   Then \( |f(z_n)| = M, 1 \text{ of } \text{cont } \implies |f(z)| = M \to z \in B. \]

\( B \) is open:
1. \( \text{Let } z \in B, D(z; R) \subseteq A. \text{ We show } D(z; R) \subseteq B: \)
   \[ \text{pf: } \text{Let } 0 < r < R. \]
   \[ \text{HVP } \implies f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) \, dt \]
   \[ \implies M = \max_{|z| = R} |f(z + re^{it})| \leq \frac{1}{2\pi} \int_0^{2\pi} M \, dt = M \]

   \[ \forall 0 < r < R, \quad 0 \leq t \leq 2\pi, |f(z + re^{it})| = M \]

   If \( |f(z + re^{it})| \) is not \( \equiv M \) \( (0 \leq t \leq 2\pi) \)

   then this is a strict inequality which would be a contradiction.

   This follows from the real analysis
   \[ \text{Lemma: Let } g(t) \leq M \text{ on } [a, b] \]
   \[ g \in C([a, b]). \]
   \[ \text{Then } \int_a^b g(t) \, dt \leq M(b - a) \]

   \[ \text{with equality iff} \]
   \[ g(t) = M, \quad a \leq t \leq b. \]

**Step 2**
\[ |f(z)| = M \implies f(z) \equiv f(z_0) \]

**pf:** (better than before).

Write \( f = u + iv. \)

Then \( |f|^2 = u^2 + v^2. \) If \( M = 0 \) then \( f(z) = 0 \) & done.

Else:
\[ 2(u \bar{u} + v \bar{v}) = 0 \]
\[ 2(u \bar{v} + v \bar{u}) = 0 \]

\[ \Rightarrow \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

Therefore matrix is singular.

Thus \( \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \) is singular.

Thus \( f(z) = \text{constant}. \)
Exercise 1: Suppose two analytic functions on $A$ are continuous on $\overline{A}$, and have the same values on $\partial A$. What can you say about the two functions? (A is bounded, open, connected)

**Soln:** Let $f$ & $g$ be these two functions.

Then $h = f - g$ is analytic. $h = 0$ on $\partial A \Rightarrow |h| = 0$ on $\partial A \Rightarrow |h| = 0$ in $A$ by maximum modulus principle. Thus $h = 0$, i.e. $f = g$.

Exercise 2: If $u$ is harmonic (and $C^2$) on $A$ and continuous on $\overline{A}$, (A is bd, open, connected) what can you say about

$$M = \max \{ u(z), z \in \overline{A} \}$$

$$m = \min \{ u(z), z \in \overline{A} \}?$$

**Ans:** $M$ & $m$ are attained on the boundary. If either is attained as $u(z_0)$ with $z_0 \in A$, then $u$ is constant!

**Proof:** In case $\exists z_0 \in A$ with $u(z_0) = M$, show $B := \{ z \in A \text{ s.t. } u(z) = M \}$ is open and closed in $A$.

$B$ closed in $A$: $\forall z_0 \in B$, $D(z_0; R) \subseteq B$.

$B$ open in $A$: let $z_0 \in B$, $D(z_0; r) \subseteq A$, while analogous ineq to page 1: $0 < r < R$.

$$M = u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} M \ d\theta = M$$

in case $\exists z_0 \in A$ with $u(z_0) = m$,

proof is same, except $> replace \leq$.

Exercise 3. The Dirichlet Problem for harmonic functions is:

Let $A \subseteq \mathbb{C}$, $\partial A$ piecewise $C^1$. Let $f: \partial A \rightarrow \mathbb{R}$ be continuous (the boundary values). Find $u \in C^2(A) \cap C(\overline{A})$ s.t.

$$\Delta u = 0 \text{ in } A$$

$$u = f \text{ on } \partial A.$$

Why can there be at most 1 solution to the Dirichlet Problem?

**Ans:** Because if $u, v$ are 2 solns, then $w := u - v$ is also harmonic and is zero on the boundary. Thus $\Delta w = 0$ in $A$ so $w = 0$; $u = v.$
Thus's an analogous formula to the Cauchy integral formula for harmonic functions.

In the case the domain is a disk it's called the Poisson integral formula:

\[ u(pe^{i\theta}) = \frac{r^2 - p^2}{2\pi} \int_0^{2\pi} \frac{u(re^{i\theta})}{r^2 - 2rp\cos(\phi - \theta) + p^2} \, d\theta \]

**Proof:** We'd like to just take the real part of C.I.F.

\[ f(z) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(t)}{t-z} \, dt \]

For \( 1 < r \), define \( z = \frac{r^2}{|z|^2} \) (or \( \frac{z}{|z|^2} \)), the "reflection" of \( z \) through the circle \( |z|=r \)

Then

\[ f(z) = \frac{1}{2\pi i} \int_{|z|=r} f(t) \left( \frac{1}{3-z} - \frac{1}{3-z^2} \right) \, dt \]

because \( I(\infty; z) = 0 \)

In fact, if

\[ u(re^{i\theta}) = f(re^{i\theta}) \]

\( 0 \leq \theta \leq 2\pi \),

this formula gives a (unique) solution to Dirichlet's problem, provided \( f \) is continuous

but, unfortunately, \( u \) & \( v \) get jumbled on RHS.

A mysterious ("reflection point") trick saves us.

For \( 1 < r \) define \( \frac{r^2}{|z|^2} \) (or \( \frac{z}{|z|^2} \)), the "reflection" of \( z \) through the circle \( |z|=r \)

\[ \frac{1}{3-z} - \frac{1}{3-z^2} = \frac{1}{3-z} - \frac{\bar{z}}{3(3-\bar{z})} \]

\[ = \frac{1}{3-z} + \frac{\bar{z}}{3(3-\bar{z})} \]

\[ = \frac{3(3-\bar{z}) + \bar{z}(3-\bar{z})}{3(3-\bar{z})^2} \]

\[ = \frac{13^2 - 13z^2}{3(3-\bar{z})^2} \]

Inseparable calculation: \( |z|=r \)

\[ \frac{1}{3-z} - \frac{1}{3-z^2} = \frac{1}{3-z} - \frac{1}{3-z^2} \]

\[ = \frac{1}{3-z} - \frac{\bar{z}}{3(3-\bar{z})} \]

\[ = \frac{1}{3-z} + \frac{\bar{z}}{3(3-\bar{z})} \]

\[ = \frac{3(3-\bar{z}) + \bar{z}(3-\bar{z})}{3(3-\bar{z})^2} \]

\[ = \frac{13^2 - 13z^2}{3(3-\bar{z})^2} \]

so, using calc. at left (and writing \( z = re^{i\theta} \))

\[ (u + iv)(z) = \frac{1}{2\pi i} \int_0^{2\pi} f(re^{i\theta}) \left( \frac{r^2 - |z|^2}{|re^{i\theta} - pe^{i\phi}|^2} \right) \, d\theta \]

Now, take real part:

\[ u(pe^{i\theta}) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{(r^2 - p^2) u(re^{i\theta})}{(re^{i\theta} - e^{i\phi})^2 + (re^{i\theta} - p\sin(\theta))^2} \, d\theta \]

assuming \( u \) harmonic on \( A \), \( D(0;r) \subset A \),

since we can construct conjugate v.

Limiting argument gives P.I.F.