Recall the mean value property for $f$ analytic on $A$, $\overline{D(z_0, R)} \subset A$:

$$ f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta $$

**Exercise 1**: Use a limiting argument to show that if $f$ is continuous on $\overline{D(z_0, R)}$ and analytic inside $D(z_0, R)$, then the mean value property still holds (more practice in exchanging limits and integrals).

**Exercise 2**: If $u(x, y)$ is harmonic in $B((x_0, y_0), R) = \{(x, y) | \sqrt{(x-x_0)^2 + (y-y_0)^2} < R\}$, if $u$ is also $C^2$ in $B$ and continues on the closed ball $\overline{B}$, use conjugate function theory to show that $u$ also satisfies the mean value property:

$$ u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{i\theta}, y_0 + Rs\sin \theta) d\theta $$
Exercise 3  Use the mean value property to prove the amazing (and surprisingly useful theoretical fact):

\textbf{maximum modulus principle:}

Let \( A \subseteq \mathbb{C} \) be bounded and open and connected.

\( f: \overline{A} \rightarrow \mathbb{C} \) continuous

\( f: A \rightarrow \mathbb{C} \) analytic.

Then \( \max \left\{ |f(z)| \text{ s.t. } z \in \overline{A} \right\} = \max \left\{ |f(z)| \text{ s.t. } z \in \partial A \right\} \)

If \( \exists z_0 \in A \text{ s.t. } |f(z_0)| \) is this maximum modulus value, then \( f(z) = f(z_0) \) is constant.

\textbf{proof:} Since \( |f(z)| \) is cont. on \( \overline{A} \), \( M = \max \left\{ |f(z)| \text{ s.t. } z \in \overline{A} \right\} \) exists (and is finite).

It suffices to show that if \( \exists z_0 \in A \text{ s.t. } |f(z_0)| = M \), then \( f \) is constant.

Let \( B = \left\{ z \in \overline{A} \text{ s.t. } f(z) = f(z_0) \right\} \). Show \( B \) is open and closed in \( A \):

Exercise 4. What is \( \max \left\{ |e^z| \text{ s.t. } z \in \overline{D(0,1)} \right\} \)?
Exercise 5: Suppose two analytic functions on $A$ are continuous on $\overline{A}$, and have the same values on $\partial A$. What can you say about the two functions? (A is bounded)

Exercise 6: If $u$ is harmonic (and $C^2$) on $A$ and continuous on $\overline{A}$, what can you say about $M = \max \{ u(z), \forall z \in \overline{A} \}$

$m = \min \{ u(z), \forall z \in \overline{A} \}$?