Some preliminary uses of the Cauchy integral formula:  \( f \) analytic in \( A, \gamma \) homotopic to a point in \( A \)

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} \, dz
\]

when \( I(\gamma, z_0) \neq 0 \) you can recover \( f(z_0) \) from the values of \( f \) along \( \gamma \)

1st application: \( \text{diff} \Rightarrow \infty \text{ly} \text{ diff} \), with estimates for derivs which only depend on \( f \):
rewrite C.I.F.

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} \, dz
\]

Theorem: \( f \) analytic in \( A \) \( \Rightarrow \) \( f \) \( \infty \text{ly} \text{ diff} \) in \( A \). In fact for \( \gamma \) homotopic to a point in \( A \),

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^n} \, dz
\]

notice, this is what we get by "differentiating thru the integral sign"

\[
\frac{d}{dz} \frac{1}{z - z_0} = \frac{-1}{(z - z_0)^2} = \frac{f(z)}{(z - z_0)^2}
\]

\[
\frac{d}{dz} \frac{1}{(z - z_0)^n} = \frac{f(z)}{(z - z_0)^{n+1}}
\]

So, when can you just do this operation?

That's an analysis question!!
analysis answer: (to justify the differentiation)

\[ G(z) = \int g(z, \bar{z}) \, d\bar{z} \]

\[ G(z+h) - G(z) = \frac{1}{h} \int g(z+h, \bar{z}) - g(z, \bar{z}) \, d\bar{z} \]

\[ \overset{?}{\Rightarrow} \int g(z, \bar{z}) \, d\bar{z} , \text{ as } h \to 0 \]

Certainly need: \( g(z, \bar{z}) \) complex differentiable in \( z \)

Then, following suffices: the difference quotients converge uniformly \( \text{wrt } z \in D(\delta) \) to \( \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \):

\[ \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t.} \]

\[ |h| < \delta \Rightarrow \left| \frac{g((z+h), \bar{z}) - g(z, \bar{z})}{h} - \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \right| < \varepsilon \]

If box holds, then

\[ |h| < \delta \Rightarrow \left| \frac{G((z+h), \bar{z}) - G(z, \bar{z})}{h} - \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \right| \leq \int_{\delta} \left| \frac{g((z+h), \bar{z}) - g(z, \bar{z})}{h} - \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \right| \, d\bar{z} \]

\[ < \varepsilon \cdot L(\delta) , \]

length

which implies

\[ G'(z) = \int \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \, d\bar{z} \]

so, how close is \( \frac{g((z+h), \bar{z}) - g(z, \bar{z})}{h} \) to \( \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \) ?

\[ \frac{1}{h} \int_{\delta} \frac{\partial g}{\partial \bar{z}} (w, \bar{z}) \, dw = \frac{1}{h} \int_{\delta} \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \, dw + \frac{1}{h} \int_{\delta} \left[ \frac{\partial g}{\partial \bar{z}} (w, \bar{z}) - \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \right] \, dw \]

\[ \overset{\text{error term}}{\left| \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \right| + \frac{1}{h} \int_{\delta} \left| \frac{\partial g}{\partial \bar{z}} (w, \bar{z}) - \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \right| \, dw \]

If \( \frac{\partial g}{\partial \bar{z}} (w, \bar{z}) \) is continuous on

\[ D(z, \delta) \times \gamma(c_1, b_1) \]

then it is uniformly continuous, so

\[ \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t.} \]

\[ |h| < \delta \Rightarrow \left| \frac{g((z+h), \bar{z}) - g(z, \bar{z})}{h} - \frac{\partial g}{\partial \bar{z}} (z, \bar{z}) \right| < \varepsilon \]

\[ \forall z \in D(\delta) \Rightarrow \text{error term} \leq |h| \varepsilon \]

\[ \text{error term} \leq |h| \varepsilon \]

[Diagram or Figure]
In our case
\[ g(z, w) = \frac{f(z)}{(3-w)^n} \]
\[ \frac{\partial g}{\partial w}(w, z) = \frac{n f(z)}{(3-w)^{n+1}} \]
is continuous on \( \overline{D(z, \varepsilon) \times \overline{D(w, \delta)}} \)
as soon as \( \varepsilon \) small enough that
\( \overline{D(z, \varepsilon)} \cap \overline{D(w, \delta)} = \emptyset \).

Two beautiful consequences of the differentiation theorem:

**Liouville's Theorem:** Let \( f: \mathbb{C} \to \mathbb{C} \) be entire (\( \mathcal{C} \)-differentiable \( \forall z \in \mathbb{C} \)) and bounded.
\[ (\exists M \in \mathbb{R} \colon |f(z)| \leq M \quad \forall z \in \mathbb{C}) \]
Then \( f \) is constant!

**Proof:** Let \( z \in \mathbb{C}, R > 0, \gamma_R(t) = z + R e^{i t}, 0 \leq t \leq 2\pi \)
so \( I(\gamma, z) = 1 \).

Thus
\[ f''(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-w)^2} \, dw \]
\[ \Rightarrow |f''(z)| \leq \frac{1}{2\pi} \int |f(z)| \, |dw| \leq \frac{1}{2\pi} \cdot \frac{M}{R^2} 
= \frac{M}{2\pi R} \leq \frac{M}{R} \]
true \( \forall R > 0, \) let \( R \to \infty \) \( \Rightarrow |f''(z)| = 0 \)
\[ \Rightarrow f \text{ constant} \]

**Fundamental Theorem of Algebra:**

Let \( p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_2 z + a_0 \) be a polynomial of degree \( n \).
(\( \text{scaled so that coeff of } z^n \text{ is } 1 \)).

Then \( p(z) \) factors into a product of \( n \) linear factors,
\[ p(z) = (z-z_1)(z-z_2)\cdots(z-z_n), \quad z_i \in \mathbb{C} \]

**Proof:**
- It suffices to prove \( \exists z \) a linear factor when \( n \geq 1 \), since general case
then follows by induction:
  - \( I) \) FTA true when \( n = 1 \)
  - \( II) \) If true for \( n-1 \), and \( p_n(z) = (z-z_n) p_{n-1}(z) \)
    then true for \( p_n(z) \).
it suffices to prove \( f \) has a root when \( n > 1 \), since if \( p_n(z_n) = 0 \),
then \( z - z_n \) is a factor of \( p_n(z) \):

\[
\frac{p_n(z)}{z - z_n} = q_{n-1}(z) + \frac{R}{z - z_n}
\]

from division algorithm.

\[
\Rightarrow p_n(z) = q_{n-1}(z)(z - z_n) + R
\]

\[
\Rightarrow p_n(z_n) = 0 + R \quad \text{in general, if} \quad p_n(z_n) = 0 \quad \text{then} \quad R = 0.
\]

So we prove that \( (\text{for } n > 1) \ p_n(z) \) has a root.
Proof is by contradiction.

If \( p(z) = p_n(z) \) does not have a root
then \( \frac{1}{p(z)} \) is entire! Now we'll show it's bounded, \( \exists M \) s.t. \( |\frac{1}{p(z)}| \leq M \)
This implies \( \frac{1}{p(z)} = \text{const} \quad \forall z \).

\[
\lim_{|z| \to \infty} \left| \frac{1}{p(z)} \right| = 0.
\]
(by Liouville, i.e. \( p(z) = \text{const} \Rightarrow \).

\[
p(z) = 2^n \left( 1 + \frac{q_{n-1}}{z^2} + \frac{q_{n-2}}{z^4} + \ldots + \frac{q_0}{z^{2n}} \right)
\]

\[
|p(z)| > |z|^{2n} \left( 1 - \left| \frac{q_{n-1}}{z^2} \right| - \left| \frac{q_{n-2}}{z^2} \right| - \ldots - \left| \frac{q_0}{z^2} \right| \right)
\]

Let \( A = \max \{ |a_{n-1}| \} \quad \forall 0 \leq i < n-1 \)

Then \( |z| \geq \max (1, 2nA) \Rightarrow \left| \frac{q_{n-1}}{z^2} \right| \leq \frac{|a_{n-1}|}{|z|} \leq \frac{A}{2nA} = \frac{1}{2n} \)

\[
\Rightarrow |p(z)| \geq |z|^{n-1} \frac{1}{2^n}
\]

\[
|p(z)| \leq \frac{2}{|z|^{n}}
\]

\[
|p(z)| \leq M \quad \forall z \in \mathcal{C}.
\]

\[
\Rightarrow \quad \text{see above!}
\]
Another consequence:

**Morera's Theorem**: Let \( f : A \to \mathbb{C} \) be continuous
and satisfy rectangle condition

\[
\oint_C f(z) \, dz = 0 \quad \forall \, R \subset A
\]

then \( f(z) \) is analytic.

**Proof**: These hypotheses guarantee \( f(z) \) has a
local complex antiderivative \( F(z) \); \( F'(z) = f(z) \)
(see b2.3 notes, sept 23)

But by the Cauchy derivative formulas, \( F(z) \)
is infinitely complex differentiable.
In particular, \( F''(z) = f'(z) \) exists!
Math 4200 Exam 1
n = 18

A

B

C