Math 4200
Monday Oct 1

Exam Wednesday, in class, closed book & note.
Format similar to practice exam.
Covers Chapters 1-2.3
Questions drawn from HW, class notes, main theorem ideas.

Review session
tomorrow (Tuesday)
8-9 a.m. LCB 222

Finish proving (Friday notes)

Index Theorem: Let $\gamma : [0,1] \to \mathbb{C} \setminus \{z_0\}$ be a p.w. C^1, closed path.

Then for a unique $n \in \mathbb{Z}$, γ is homotopic to $\gamma^n(t) = z_0 + e^{2\pi i n t}$, $0 \le t \le 1$

and so

$$I(\gamma; z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{\gamma^{(n)}} \frac{1}{z-z_0}\,dz = n$$

"index of γ w.r.t z_0", or "winding number of γ about z_0"

(in practice $I(\gamma; z_0)$ can be computed without reparameterizing γ so that its domain is $[0,1]$)

The definition of index and the deformation theorem leads to the following result, which will underlie almost every other result we prove in this course.
Notice how amazing this result is!

Cauchy Integral formula Let A be open and simply connected
$f : A \to \mathbb{C}$ analytic.
$z_0 \in A$
γ a p.w. C^1 closed contour in A, with z_0 not on the contour

Then $$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)\, I(\gamma; z_0)$$

$f(z_0)$ is determined by the values of $f(z)$ on γ!
(as long as $I(\gamma; z_0) \neq 0$).

<u>proof</u>: Define $g(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} & z \neq z_0 \\ f'(z_0) & z = z_0 \end{cases}$

g is analytic in $A \setminus \{z_0\}$, and continuous at z_0.

If we can show the rectangle lemma for g (see below*), then
local antiderivative theorem follows (just used rect. lemma + g continuous)
so, deformation theorem does too.
In particular, g has a global antideriv. G on simply-connected A,
So

$$\int_\gamma g(z)\,dz = 0$$

$$\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = \int_\gamma \frac{f(z)}{z-z_0}\,dz - f(z_0)\underbrace{\int_\gamma \frac{1}{z-z_0}\,dz}_{2\pi i\, I(\gamma; z_0)}$$

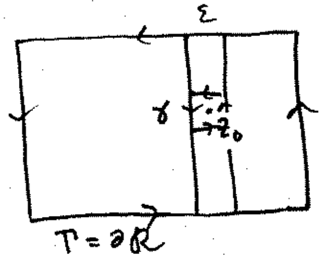
i.e.

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)\, I(\gamma; z_0) \quad \blacksquare$$

*rectangle lemma for g: let $R = \{x+iy,\ a \le x \le b,\ c \le y \le d\} \subset A$

if $z_0 \notin R$ $\int_{\partial R} g(z)\,dz = 0$ as before.

if $z_0 \in$ interior of R:



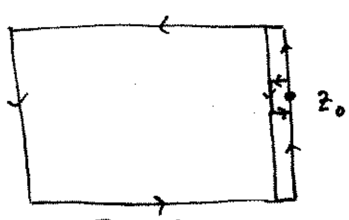
$$\int_\Gamma g(z)\,dz = \int_\gamma g(z)\,dz$$ by rectangle lemmas and contour integral cancellation.

If γ is the bdry of a square with side-lengths ε, then

$$\left|\int_\gamma g(z)\,dz\right| \le \int_\gamma |g(z)|\,|dz| \le 4\varepsilon M$$ where $M = \max_R |g(z)|$.

$$\varepsilon \to 0 \Rightarrow \int_\Gamma g(z)\,dz = 0$$

if $z_0 \in \partial R$,
argument is same as interior case, except use



(or corner picture)